## Linear Momentum

We have already studied about the newton's laws of motion and about their application
It becomes difficult to use Newton's law of motion as it is while studying complex problems like collision of two objects, motion of the molecules of the gas,rocket propulsion system etc
Thus a further study of newton's law is required to find some theorem or principles which are direct consequences of Newton's law
We have already studied one such principle which is principle of conservation of energy. Here in this chapter we will define momentum and learn about the principle of conservation of momentum .
Thus we begin this chapter with the concept of impluse and momentum which like work and energy are developed from Newton's law of motion

## (2) Impulse and momentum

To explain the terms impulse and momentum consider a particle of mass $m$ is moving along $x$-axis under the action of constant force F as shown below in the figure


Figure 1
If at time $t=0$, velocity of the particle is $v_{0}$ then at any time $t$ velocity of particle is given by the equation $\mathbf{v}=\mathbf{v}_{0}+\mathbf{a t}$
where $\mathbf{a}=\mathrm{F} / \mathrm{m}$
can be determined from the newton's second law of motion .Putting value of acceleration in above equation we get
$m v=m v_{0}+F t$
or
$\mathrm{Ft}=\mathrm{mv}-\mathrm{mv}_{0}$
right side of the equation Ft , is the product of force and the time during which the force acts and is known as the impluse

Thus
Impulse= Ft
If a constant force acts on a body during a time from $t_{1}$ and $t_{2}$, then impulse of the force is
I $=F\left(\mathrm{t}_{2}-\mathrm{t}_{1}\right)$
Thus impulse recieved during an impact is defined as the product of the force and time interval during which it acts
Again consider left hand side of the equation (1) which is the difference of the product of mass and velocity of the particle at two different times $\mathrm{t}=0$ and $\mathrm{t}=\mathrm{t}$
This product of mass and velocity is known as linear momentum and is represented by the symbol $\mathbf{p}$.
Mathematically
$\mathbf{p}=\mathrm{mv}$
physically equation (1) states that the impulse of force from time $t=0$ to $t=t$ is equal to the change in linear momentum during
If at time $t_{1}$ velocity of the particle is $v_{1}$ and at time $t_{2}$ velocity of the particle is $v_{2}$, then
$F\left(t_{2}-t_{1}\right)=m v_{2}-m v_{1}$
so far we have considered the case of the particle moving in a straight line i.e along $x$-axis and quantities
involved F,v, and a were all scalars
If we call these quantities as components of the vectors $\mathbf{F}, \mathbf{v}$ and $\mathbf{a}$ along $\mathbf{x}$-axis and generalize the definations of momentum and impulse so that the motion now is not constrained along one -direction ,Thus we got

Impulse $=I=F\left(t_{2}-t_{1}\right)$
Linear momentum=p=mv
where
$\mathrm{I}=\mathrm{I}_{\mathrm{x}} \mathrm{i}+\mathrm{l}_{\mathrm{y}} \mathrm{j}+\mathrm{l}_{\mathrm{z}} \mathbf{k}$
$F=F_{x} i+F_{y} j+F_{z} k$
$p=p_{x} i+p_{y} j+p_{z} k$
$\mathbf{v}=\mathbf{v}_{\mathrm{x}} \mathbf{i}+\mathrm{v}_{\mathrm{y}} \mathbf{j} \mathbf{j}+\mathrm{v}_{\mathrm{z}} \mathbf{k}$
are expressed in terms of their components along $x, y$ and $z$ axis and also in terms of unit vectors
On generalizing equation (4) using respective vectors quantities we get the equation
$F\left(\mathrm{t}_{2}-\mathrm{t}_{1}\right)=\mathrm{m} \mathbf{v}_{2}-\mathrm{m} \mathbf{v}_{1}$

So far while discussing Impulse and momentum we have considered force acting on particle is constant in direction and maagnitude
In general ,the magnitude of the force may vary with time or both the direction and magnitude may vary with time
Consider a particle of mass m moving in a three-dimensional space and is acted upon by the varying resultant force $\mathbf{F}$. Now from newtons second law of motion we know that
$\mathrm{F}=\mathrm{m}(\mathrm{d} \mathbf{v} / \mathrm{dt})$
or $\mathbf{F d t}=\mathrm{mdv}$
If at time $t_{1}$ velocity of the particle is $\mathbf{v}_{1}$ and at time $t_{2}$ velocity of the particle is $\mathbf{v}_{2}$, then from above equation we have
$\int_{t_{1}}^{t_{2}} \mathbf{F} d t=\int_{\mathrm{v}_{1}}^{\mathrm{v}_{2}} m d \mathbf{v}$
Integral on the left hand side of the equation (8) is the impulse of the force $\mathbf{F}$ in the time interval $\left(\mathrm{t}_{2}-\mathrm{t}_{1}\right)$ and is a vector quantity,Thus
Impulse $=\mathrm{I}=\int_{t^{2}}^{t_{5}} \mathrm{~F} d t$
Above integral can be calculated easily if the Force $F$ is some known function of time $t$ i.e.,

Integral on the right side is when evaluated gives the product of the mass of the particle and change in the velocity of the partcile
$\int_{\mathbf{v}_{1}}^{\mathbf{v}_{2}} m d \mathbf{v}=m \int_{\mathbf{v}_{1}}^{\mathbf{v}_{2}} d \mathbf{v}=m\left(\mathbf{v}_{2}-\mathbf{v}_{1}\right)$
using equation (9) and (10) to rewrite the equation (8) we get
$\int_{t_{1}}^{t_{2}} \mathbf{F} d t=m\left(\mathbf{v}_{2}-\mathbf{v}_{1}\right)$
Equivalent equations of equation (11) for particle moving in space are
$\int_{\hbar}^{t_{t}} F_{x} d t=m v_{x 2}-m v_{x 1}$
$\int_{\hbar}^{t_{2}} F_{y} d t=m v_{y 2}-m v_{y 1}$
$\int_{\hbar}^{t} F_{z} d t=m v_{z 2}-m v_{z 1}$


Thus we conclude that impulse of force $\mathbf{F}$ during the time interval $\mathrm{t}_{2}-\mathrm{t}_{1}$ is equal to the change in the linear momentum of the body on which its acts
SI units of impulse is Ns or Kgms_1

## (3) Conservation of Linear momentum

Law of conservation of linear momentum is a extremely important consequence of Newton's third law of motion in combination with the second law of motion
Consider two particles of mass $\mathrm{m}_{1}$ and $\mathrm{m}_{2}$ interacting with each other and forces acting on these particles are only the ones they exert on each other.
Let $F_{12}$ be the force exerted by the particle 2 on particle 1 having mass $m_{1}$ and velocity $\mathbf{v}_{1}$ and $F_{21}=-F_{12}$ be the force exerted by the particle 1 on particle 2 having mass $m_{2}$ and velocity $\mathbf{v}_{2}$

Applying newton second law of each particle on each partcile

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\(\mathrm{F}_{12}=\mathrm{m}_{1}\left(\mathrm{~d} \mathrm{v}_{1} / \mathrm{dt}\right)\)
and \(F_{21}=m_{2}\left(d v_{2} / d t\right)\)
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from newton's third law of motion
$F_{21}=-F_{12}$
or $m_{1}\left(d v_{1} / d t\right)+m_{2}\left(d v_{2} / d t\right)=0$
Since mass of the particles are not varying with time so we can write $(\mathrm{d} / \mathrm{dt})\left(\mathrm{m}_{1} \mathrm{v}_{1}+\mathrm{m}_{2} \mathrm{v}_{2}\right)=0$
or $m_{1} \mathrm{v}_{1}+\mathrm{m}_{2} \mathrm{v}_{2}=$ constant
we have already defined the quantity mv as the momentum of the particle
Thus we conclude that when two particles are subjected only to their mutual interactions ,the sum of the momentums of the bodies remains constant in time or we can say the total momentum of the two particles does not change becuase of the any mutual interactions between them
For any kind of force between two particles then sum of the momentum ,both before and after the action of force should be equal i.e total momentum remains constant

We thus arrive to the statment of principle of conservation of linear momentum
" when no resultant external force acts on system ,the total momentum of the system remains constant in magnitude and direction"

In absence of external forces for a number of interacting particles, law of conservation of linear momentum can be expressed as
$m_{1} \mathrm{v}_{1}+\mathrm{m}_{2} \mathbf{v}_{2}+\mathrm{m}_{3} \mathbf{v}_{3}+\mathrm{m}_{4} \mathbf{v}_{4}+\ldots=$ constant
Law of conservation of linear momentum is one of the most fundamental and important principle of mechanics This principle also holds true even if the forces between the interacting particles is not conservative
Once again ,the total momentum of two or any number of particles of interacting particles is constant if they are isolated
from outside influences (or no resultant external forces is acting on the particles)

## (4) Recoil of a gun

Consider the gun and bullet in its barrel as an isolated system
In the begining when bullet is not fired both the gun and bullet are at rest.So the momentum of the before firing is zero
$\mathrm{p}_{\mathrm{i}}=0$
Now when the bullet is fired, it moves in the forward direction and gun recoil back in the opposite direction Let $m_{b}$ be the mass and $v_{b}$ of velocity of the bullet And $m_{g}$ and $v_{g}$ be the velcoity of the gun after the firing Total momentum of the system after the firing would be
$\mathrm{p}_{\mathrm{f}}=\mathrm{m}_{\mathrm{b}} \mathrm{v}_{\mathrm{b}}+\mathrm{m}_{\mathrm{g}} \mathrm{v}_{\mathrm{g}}$
since no external force are acting on the system, we can apply the law of conservation of linear momentum to the system
Therfore
$\mathrm{p}_{\mathrm{f}}=\mathrm{p}_{\mathrm{i}}$
or $m_{b} v_{b}+m_{g} v_{g}=0$
or $\mathrm{v}_{\mathrm{g}}=-\left(\mathrm{m}_{\mathrm{b}} \mathrm{v}_{\mathrm{b}} / \mathrm{m}_{\mathrm{g}}\right)$

The negative sign in above equation shows that velocity of the recoil of gun is opposite to the velocity of the bullet

Since mass of the gun is very large as compared to the mass of the bullet,the velocity of the recoil is very small as compared to the velocity of the bullet
(5) Motion of the system with varying mass(Rocket)

Uptill now while studying classical mechanics we have always considered the particle under consideration to have constant mas
Some times it is required to deal with the particles or system of particles in which mass is varying and motion of the rocket is one such examples
In a rocket fuel is burned and the exhaust gas is expelled out from the rear of the rocket
The force exerted by the exhaust gas on the rocket is equal and opposite to the force exerted by the rocket to expell it
This force exerted by the exhaust gas on the rocket propels the rocket forwards
The more gass is ejected from the rocket ,the mass of the rcoket decreases


Figure 2a. Rocket at time t after takeoff with mass $m$ and velocity vin upwards direction


Figure 2b. Rocket at time $t+\Delta t$ after takeoff with mass $\mathrm{m}+\mathrm{dm}$ and velocity $\mathrm{v}+\mathrm{dv}$ in upwards direction

To analyze this process let us consider a rocket being fired in upwards direction and we neglect the resistance offered by the air to the motion of the rocket and variation in the value of the acceleration due to gravity with height
Figure above shows a rocket of mass $m$ at a time $t$ after its take off moving with velocity $v$. Thus at time $t$ momentum of the rocket is equal to mv.THus
$p_{i}=m v$
Now after a short interval of time dt,gas of total mass dm is ejected from the rocket
If $v_{g}$ represents the downward speed of the gas relative to the rocket then velocity of the gas relative to earth is $v_{g e}=v-v_{g}$
And its momentum equal to
$\mathrm{dmv}_{\mathrm{ge}}=\mathrm{dm}\left(\mathrm{v}-\mathrm{v}_{\mathrm{g}}\right)$
At time $t+d t$,the rocket and unburned fuel has mass $m-d m$ and its moves with the speed $v+d v$.Thus momentum
of thee rocket is $=(m-d m)(v+d v)$
Total momentum of the system at time $t+d t$ is
$p_{f}=d m\left(v-v_{g}\right)+(m-d m)(v+d v)$
Here system constitute the ejected gas and rocket at the time $t+d t$
From Impulse momentum relation we know that change in momentum of the system is equal to the product of the resultant external force acting on the system and the time interval during which the force acts Here external force on the rocket is weight -mg of the rocket ( the upward direction is taken as positive)
Now
Impulse=change in momentum
$F_{e x t} d t=p_{f}-p_{i}$
or
$-m g d t=d m\left(v-v_{g}\right)+(m-d m)(v+d v)-m v$
or
$-m g d t=m d v-v_{g} d m-d m d v$
term dmdv can be dropped as this product is neglibigle in comparison of other two terms Thus we have
$-m g=m \frac{d v}{d t}-v_{g} \frac{d m}{d t}$
or
$m \frac{d v}{d t}=v_{g} \frac{d m}{d t}-m g$
In equation (14) dv/dt represent the acceleration of the rocket,so mdv/dt =resulant force on the rocket

## Therefore

Resultant Force on rocket=Upthrust on the rocket - weight of the rocket where upthrust on rocket $=\mathrm{v}_{\mathrm{g}}(\mathrm{dm} / \mathrm{dt})$

The upthrust on rocket is proportional to both the relative velocity $\left(v_{g}\right)$ of the ejected gas and the mass of the gas ejected per unit time (dm/dt)
Again from equation (14)
$\frac{d v}{d t}=\frac{v_{g}}{m} \frac{d m}{d t}-g$
As rocket goes higher and higher, value of the acceleration due to gravity $g$ decreases continously .The values of $v_{g}$ and $d m / d t$ parctically remains constant while fuel is being consumed but remaining mass $m$ decreases continously. This result in continous increase in acceleration of rocket untill all the fuel is burned up Now we will find the relation between the velocity at any time $t$ and remaining mass.Again from equation (15) we have
$d v=v_{g}(d m / m)-g d t$
Here dm is a +ve quantity representing mass ejected in time dt . So change in mass of the rocket in time dt is dm . So while calculating total mass change in rocket, we must change the sign of the term containing dm $d v=-v_{g}(d m / m)-g d t$
Initially at time $t=0$ if the mass and velocity of the rocket are $m_{0}$ and $v_{0}$ respectively.After time $t$ if $m$ and $v$ are mass and velocity of the rocket then integrating equation (16) within these limits
$\int_{v_{0}}^{v} d v=\int_{m_{0}}^{v}-v_{g} \frac{d m}{m}-\int_{0}^{t} g d t$
On evaluating this integral we get
$v-v_{0}=-v_{g}\left(\ln m-\ln m_{0}\right)-g(t-0)$
or $v=v_{0}+v_{g} \ln \left(m_{0} / m\right)-g t$
equation(17) gives the change in velocity of the rocket in terms of exhaust speed and ration of initial amd final masses at any time t
The speed acquired by the rocket when the whole of the fuel is burned out is called burn-out speed of the rocket

System of particles and Collision

## (1) Introduction

Until now we have focused on describing motion of a single particle in one, two or three dimensions. By particle we mean to say that it has a size negligible in comparison to the path raveled by it.
When we studied Law's of motion we have applied them even to the bodies having finite size imagining that motion of such bodies can be described in terms of motion of particles.
While doing so we have ignored the the internal structure of such bodies. Any real body we encounter in our daily life has a finite size and idealized model of particle is inadequate when we deal with motion of real bodies of finite size .
Real bodies of finite size can also be regarded as the system of particles. While studying system of particles we will not concentrate on each and every particle of the system instead we will consider the motion of system as a whole.
Large number of problems involving extended bodies or real bodies of finite size can be solved by considering them as Rigid Bodies. We define rigid body as a body having definite and unchanging shape.
A rigid body is a rigid assembly of particles with fixed inter-particle distances. In actual bodies deformations do occur but we neglect them for the sake of simplicity.
In this chapter we will study about centre of mass of system of particles, motion of centre of mass and about collisions.

## (2) Centre of mass

Consider a body consisting of large number of particles whose mass is equal to the total mass of all the particles. When such a body undergoes a translational motion the displacement is produced in each and every particle of the body with respect to their original position.
If this body is executing motion under the effect of some external forces acting on it then it has been found that there is a point in the system, where if whole mass of the system is supposed to be concentrated and the nature the motion executed by the system remains unaltered when force acting on the system is directly applied to this point. Such a point of the system is called centre of mass of the system.
Hence for any system Centre of mass is the point where whole mass of the system can be supposed to be concentrated and motion of the system can be defined in terms of the centre of mass.

Consider a stationary frame of refrance where a body of mass $M$ is situated. This body is made up of $n$ number of particles. Let $m$ be the mass and $\mathbf{r}$ be the pisition vector of i'th particle of the body.

Let $C$ be any point in the body whose position vector with respect to origin $O$ of the frame of refrance is $R_{C}$ and position vector of point C w.r.t. i'th particle is $\mathbf{r}_{\mathrm{ci}}$ as shown below in the figure.


Figure 1. $C$ is the position of center of mass the rigid body of mass $M$

From triangle OCP
$\mathbf{r}_{\mathrm{i}}=\mathbf{R}_{\mathrm{C}}+\mathbf{r}_{\mathrm{ci}}$
multiplying both sides of equation 1 bt $m_{i}$ we get
$\mathrm{m}_{\mathrm{i}} \mathbf{r}_{\mathrm{i}}=\mathrm{m}_{\mathrm{i}} \mathbf{R}_{\mathrm{C}}+\mathrm{m}_{\mathbf{i}} \mathbf{r}_{\mathrm{ci}}$
taking summation of above equation for n particles we get

$$
\begin{equation*}
\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{R}_{\mathrm{c}}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{ci}} \tag{2}
\end{equation*}
$$

If for a body
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{ci}}=0$
then point C is known as the centre of mass of the body.
Hence a point in a body w.r.t. which the sum of the product of mass of the particle and their position vector is equal to zero is equal to zero is known as centre of mass of the body.

## (3) Position of centre of mass

## (i) Two particle system

Consider a system made up of two particles whose mass are $\mathrm{m}_{1}$ and $\mathrm{m}_{2}$ and their respective position vectors w.r.t. origin $O$ be $\mathbf{r}_{1}$ and $\mathbf{r}_{2}$ and $\mathbf{R}_{\mathrm{cm}}$ be the position vector of centre of mass of the system as shown below in the figure. So from equation 2


Figure 2. Position of center of mass for a two particle system
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{ci}}=0$

## hence

$\sum_{\mathrm{i}=1}^{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}=\sum_{\mathrm{i}=1}^{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{R}_{\mathrm{cm}}=\boldsymbol{R}_{\mathrm{cm}} \sum_{\mathrm{i}=1}^{2} \mathrm{~m}_{\mathrm{i}}$
or,
$\boldsymbol{R}_{\mathrm{cm}} \sum_{\mathrm{i}=1}^{2} \mathrm{~m}_{\mathrm{i}}=\sum_{\mathrm{i}=1}^{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}$
or,
$\boldsymbol{R}_{\mathrm{cm}}=\frac{\sum_{\mathrm{i}=1}^{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}}{\sum_{\mathrm{i}=1}^{2} \mathrm{~m}_{\mathrm{i}}}$
$\boldsymbol{R}_{\mathrm{cm}}=\frac{\mathrm{m}_{1} \boldsymbol{r}_{1}+\mathrm{m}_{2} \boldsymbol{r}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}$
If $M=m_{1}+m_{2}=$ total mass of the system, then

$$
\begin{align*}
& \boldsymbol{R}_{\mathrm{cm}}=\frac{\mathrm{m}_{1} \boldsymbol{r}_{1}+\mathrm{m}_{2} \boldsymbol{r}_{2}}{\mathrm{M}} \\
& \text { or, }  \tag{5}\\
& \mathrm{M} \boldsymbol{R}_{\mathrm{cm}}=\mathrm{m}_{1} \boldsymbol{r}_{1}+\mathrm{m}_{2} \boldsymbol{r}_{2}
\end{align*}
$$



## (ii) Many particle system

Consider a many particle system made up of number of particles as shown below in the figure. Let $\mathrm{m}_{1}, \mathrm{~m}_{2}$, $m_{3}, \ldots \ldots \ldots, m_{n}$ be the masses of the particles of system and their respective position vectors w.r.t. origin are $\mathbf{r}_{1}, \mathbf{r}_{2}, \mathbf{r}_{3}, \ldots \ldots \ldots \ldots .$.


Figure 3. Position of center of mass for many particle system
Also position vector of centre of mass of the system w.r.t. origin of the reference frame be $\mathbf{R}_{\mathrm{cm}}$ then from equation 2
$\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{r}_{\boldsymbol{i}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{R}_{\boldsymbol{c m}}$
Because of the definition of centre of mass

$$
\begin{align*}
& \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{ci}}=0 \\
& \text { or, } \\
& \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{R}_{\mathrm{cm}}=\boldsymbol{R}_{\mathrm{cm}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \\
& \text { or, } \\
& \boldsymbol{R}_{\mathrm{cm}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}} \\
& \text { or, } \\
& \boldsymbol{R}_{\mathrm{cm}}=\frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}}{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}} \\
& \boldsymbol{R}_{\mathrm{cm}}=\frac{\mathrm{m}_{1} \boldsymbol{r}_{1}+\mathrm{m}_{2} \boldsymbol{r}_{2}+\mathrm{m}_{3} \boldsymbol{r}_{3}+\ldots \ldots \ldots . .+\mathrm{m}_{\mathrm{n}} \boldsymbol{r}_{n}}{\mathrm{~m}_{1}+\mathrm{m}_{2}+\mathrm{m}_{3}+\ldots \ldots \ldots .+\mathrm{m}_{\mathrm{n}}}  \tag{7}\\
& \boldsymbol{R}_{\mathrm{cm}}=\frac{\mathrm{m}_{1} \boldsymbol{r}_{1}+\mathrm{m}_{2} \boldsymbol{r}_{2}+\mathrm{m}_{3} \boldsymbol{r}_{3}+\ldots \ldots \ldots .+\mathrm{m}_{\mathrm{n}} \boldsymbol{r}_{n}}{\mathrm{M}_{\mathrm{M}}} \tag{8}
\end{align*}
$$

where,
$\mathrm{M}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}}=$ total mass of the system

## (4) Position vector of centre of mass in terms of co-ordinate components

Let in a system of many particles the co-ordinates of centre of mass of the system be $\left(\mathrm{X}_{\mathrm{cm}}, \mathrm{Y}_{\mathrm{cm}}, \mathrm{Z}_{\mathrm{cm}}\right)$ then position vector of centre of mass would be
$\mathbf{R}_{\mathrm{cm}}=\mathrm{X}_{\mathrm{cm}}{ }^{\mathrm{i}+\mathrm{Y}_{\mathrm{cm}}{ }^{\mathbf{j}}+\mathrm{Z}_{\mathrm{cm}} \mathbf{k}, ~}$
and position vectors of various particles would be
EQ
Putting the values from equation 9 and 10 in equation 6, 7 and 8 we get

$$
\begin{align*}
& \boldsymbol{r}_{1}=\mathrm{x}_{1} \boldsymbol{i}+\mathrm{y}_{1} \boldsymbol{j}+\mathrm{z}_{1} \boldsymbol{k} \\
& \boldsymbol{r}_{2}=\mathrm{x}_{2} \boldsymbol{i}+\mathrm{y}_{2} \boldsymbol{j}+\mathrm{z}_{3} \boldsymbol{k} \\
& \boldsymbol{r}_{3}=\mathrm{x}_{3} \boldsymbol{i}+\mathrm{y}_{3} \boldsymbol{j}+\mathrm{z}_{3} \boldsymbol{k} \tag{10}
\end{align*}
$$

$\boldsymbol{r}_{\boldsymbol{n}}=\mathrm{x}_{\mathrm{n}} \boldsymbol{i}+\mathrm{y}_{\mathrm{n}} \boldsymbol{j}+\mathrm{z}_{\mathrm{n}} \boldsymbol{k}$
If in any system there are infinite particles of point mass and are distributed continously also if the distance between them is infinitesimally small then summation in equations $6,11,12$ and 13 can be replaced by integration. If $r$ is the position vector of very small particle of mass $d m$ of the system then position vector of centre of the system would be

$$
\begin{align*}
& \mathrm{x}_{\mathrm{cm}}=\frac{\mathrm{m}_{1} \mathrm{x}_{1}+\mathrm{m}_{2} \mathrm{x}_{2}+\mathrm{m}_{3} \mathrm{x}_{3}+\ldots \ldots \ldots .+\mathrm{m}_{\mathrm{n}} \mathrm{x}_{n}}{\mathrm{~m}_{1}+\mathrm{m}_{2}+\mathrm{m}_{3}+\ldots \ldots \ldots+\mathrm{m}_{\mathrm{n}}}=\frac{1}{\mathrm{~m}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \mathrm{x}_{\mathrm{i}}  \tag{11}\\
& \mathrm{Y}_{\mathrm{cm}}=\frac{\mathrm{m}_{1} \mathrm{y}_{1}+\mathrm{m}_{2} \mathrm{y}_{2}+\mathrm{m}_{3} \mathrm{y}_{3}+\ldots \ldots \ldots .+\mathrm{m}_{\mathrm{n}} \mathrm{y}_{n}}{\mathrm{~m}_{1}+\mathrm{m}_{2}+\mathrm{m}_{3}+\ldots \ldots \ldots .+\mathrm{m}_{\mathrm{n}}}=\frac{1}{\mathrm{~m}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \mathrm{y}_{\mathrm{i}}  \tag{12}\\
& \mathrm{Z}_{\mathrm{cm}}=\frac{\mathrm{m}_{1} \mathrm{z}_{1}+\mathrm{m}_{2} \mathrm{z}_{2}+\mathrm{m}_{3} \mathrm{z}_{3}+\ldots \ldots \ldots .+\mathrm{m}_{\mathrm{n}} \mathrm{z}_{n}}{\mathrm{~m}_{1}+\mathrm{m}_{2}+\mathrm{m}_{3}+\ldots \ldots \ldots+\mathrm{m}_{\mathrm{n}}}=\frac{1}{\mathrm{~m}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \mathrm{z}_{\mathrm{i}} \tag{13}
\end{align*}
$$

and the value of its co-ordinates would be
$\mathbf{R}_{c m}=\frac{1}{M} \int \mathbf{r} d m$
and the values of its co-ordinates would be
$\mathrm{X}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \int \mathrm{xdm}$
$\mathrm{Y}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \int \mathrm{ydm}$
$\mathrm{Z}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \int \mathrm{zdm}$
If $\rho$ is the density of the system then $d m=\rho d V$ where $d V$ is the very small volume element of the system then,
$\boldsymbol{R}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \int \rho \boldsymbol{r} \mathrm{dV}$
$\mathrm{X}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \int \rho \mathrm{xdV}$
$\mathrm{Y}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \int \rho \mathrm{ydV}$
$Z_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \int \rho \mathrm{zdV}$
The centre of mass of a homogeneous body (body having uniform distribution of mass) must concide with the geometrical centre of the body. In other words we can say that if the homogeneous body has a point, a line or plane of symmetry, then its centre of mass must lie at this point , line or plane of symmetry.
The centre of mass of irregular bodies and shape can be found using equations $14,15,16$.

## (5) Motion of centre of mass

Differentiating equation 6 we get
$\frac{\mathrm{d} \boldsymbol{R}_{\mathrm{cm}}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}}{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}}}\right)=\frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \frac{\mathrm{d} \boldsymbol{r}_{\mathrm{i}}}{\mathrm{dt}}}{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}}}$
but, $\frac{\mathrm{d} \boldsymbol{R}_{\mathrm{cm}}}{\mathrm{dt}}=\boldsymbol{V}_{\mathrm{cm}}$ which is the velocity of centre of mass
$\frac{\mathrm{d} \boldsymbol{r}}{\mathrm{dt}}=\boldsymbol{v}_{\mathrm{i}}$ is the velocity of $\mathrm{i}^{\prime}$ th particle of the system
Therefore
$\boldsymbol{V}_{\mathrm{cm}}=\frac{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{v}_{i}}{\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}}}=\frac{1}{\mathrm{M}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{v}_{i}$
but, $\mathrm{m}_{\mathrm{i}} \boldsymbol{v}_{\boldsymbol{i}}=\boldsymbol{p}_{\boldsymbol{i}}$ which is the linear momentum of the i 'th particle of the system

Therefore $\quad \boldsymbol{V}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \sum_{\mathrm{i}=1}^{\mathrm{n}} p_{i}=\frac{\boldsymbol{p}}{\mathrm{M}}$
Or, $\quad \mathrm{M} \boldsymbol{V}_{\mathrm{cm}}=\boldsymbol{p}$
where $\boldsymbol{p}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \boldsymbol{p}_{\mathrm{i}}$
$\mathbf{p}$ is the vector sum of linear momentum of various particles of the system or it is the total linear momentum of the system.
If no external force is acting on the system then its linear momentum remains constant. Hence in absence of external force
$\mathrm{M} V_{\mathrm{cm}}=$ constan t
or,
$V_{\mathrm{cm}}=$ constant


In the absence of external force velocity of centre of mass of the system remains constant or we can say that centre of mass moves with the constant velocity in absence of external force.
Hence from equation 18 we came to know that the total linear momentum of the system is equal to the product of the total mass of the system and the velocity of the centre of mass of the system which remains constant. Thus in the absence of the external force it is not necessary that momentum of individual particles of system like $\mathbf{p}_{1}, \mathbf{p}_{2}, \mathbf{p}_{3} \ldots \ldots \ldots \mathbf{p}_{\mathrm{n}}$ etc. remains constant but their vector sum always remains constant.

## (6) Acceleration of centre of mass

Differentiating equation 7 we get
$\frac{\mathrm{d} \boldsymbol{V}_{\mathrm{cm}}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{\mathrm{M}} \sum_{i=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{v}_{i}\right]=\frac{1}{\mathrm{M}} \sum_{i=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \frac{\mathrm{d} \boldsymbol{v}_{i}}{\mathrm{dt}}$
$\frac{\mathrm{d} \boldsymbol{v}_{i}}{\mathrm{dt}}=\boldsymbol{a}_{i}=$ acceleration of the i'th particle
$\frac{\mathrm{d} \boldsymbol{V}_{\mathrm{cm}}}{\mathrm{dt}}=\boldsymbol{a}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \sum_{i=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}} \boldsymbol{a}_{i}=$ acceleration of centre of mass of the system
If,
$\mathrm{m}_{\mathrm{i}} \boldsymbol{a}_{i}=\boldsymbol{F}_{\mathrm{i}}$ which is the force acting on the i 'th particle of the system then,
$\boldsymbol{a}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \sum_{i=1}^{\mathrm{n}} \boldsymbol{F}_{i}$
If $m_{i} \mathbf{a}_{i}=F_{i}$ which is the force acting on the i'th particle of the system then
$\boldsymbol{a}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \sum_{i=1}^{\mathrm{n}}\left[\boldsymbol{F}_{\mathrm{i}(\mathrm{ext})}+\boldsymbol{F}_{\mathrm{i}(\mathrm{nt})}\right]$
Net force acting on the i'th is
$\mathrm{F}_{\mathrm{i}}=\mathrm{F}_{\mathrm{i}(\mathrm{ext})}+\mathrm{F}_{\mathrm{i}(\text { (int })}$
Here internal force is produced due to the mutual interaction between the particles of the system. Therefore, from Newton's third law of motion

$$
\begin{aligned}
& \boldsymbol{F}_{\mathrm{ij}}=-\boldsymbol{F}_{\mathrm{ji}} \\
& \therefore \sum_{\mathrm{i}=1}^{n} \boldsymbol{F}_{\mathrm{ij}}=0 \\
& \text { hence, } \\
& \boldsymbol{a}_{\mathrm{cm}}=\frac{1}{\mathrm{M}} \sum_{i=1}^{\mathrm{n}} \boldsymbol{F}_{\mathrm{i}(e x t)}=\frac{\boldsymbol{F}}{\mathrm{M}} \\
& \text { where, } \\
& \boldsymbol{F}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \boldsymbol{F}_{i}
\end{aligned}
$$

is the total external force acting on the system since internal force on the system because of mutual interaction between the particles of the system become equal to zero because of the action reaction law.
Hence from equation 23 it is clear that the centre of mass of the system of particles moves a if the whole mass of the system were concentrated at it. This result holds whether the system is a rigid body with particles in fixed position or system of particles with internal motions.

## (7) Kinetic energy of the system of particles

Let there are n number of particles in a n particle system and these particles possess some motion. The motion of the i'th particle of this system would depend on the external force $\mathbf{F}_{\mathrm{i}}$ acting on it. Let at any time if the velocity of $i$ 'th particle be $\mathbf{v}_{\mathrm{i}}$ then its kinetic energy would be

$$
\begin{align*}
& \mathrm{E}_{\mathrm{Ki}}=\frac{1}{2} \mathrm{mv}_{\mathrm{i}}^{2} \\
& \mathrm{E}_{\mathrm{Ki}}=\frac{1}{2} \mathrm{~m}\left(\boldsymbol{v}_{\mathrm{i}} \cdot \boldsymbol{v}_{\mathrm{i}}\right) \tag{1}
\end{align*}
$$

Let $\mathbf{r}_{\mathrm{i}}$ be the position vector of the $i^{\prime \prime}$ th particle w.r.t. $O$ and $\mathbf{r}^{\prime}$; be the position vector of the centre of mass w.r.t. $\mathbf{r}_{\mathrm{i}}$ ,as shown below in the figure , then
$\mathbf{r}_{\mathrm{i}}=\mathbf{r}^{\prime}{ }_{i}+\mathbf{R}_{\mathrm{cm}}$
where $\mathbf{R}_{\mathrm{cm}}$ is the position vector of centre of mass of the system w.r.t. O.

Figure 4. $\boldsymbol{r}_{i}^{\prime}$ Is the position vector of center of mass w.r.t. $\boldsymbol{r}_{i}$ Differentiating equation 2 we get

$$
\frac{\mathrm{d} \boldsymbol{r}_{\mathrm{i}}}{\mathrm{dt}}=\frac{\mathrm{d} \boldsymbol{r}_{\mathrm{i}}^{\prime}}{\mathrm{dt}}+\frac{\mathrm{d} \boldsymbol{R}_{\mathrm{cm}}}{\mathrm{dt}}
$$

or,
$\boldsymbol{v}_{\mathrm{i}}=\boldsymbol{v}_{\mathrm{i}}^{\prime}+\boldsymbol{V}_{\mathrm{cm}}$
where $\mathbf{v}_{\mathrm{i}}$ is the velocity of i 'th particle w.r.t. centre of mass and $\mathbf{V}_{\mathrm{cm}}$ is the velocity of centre of mass of system of particle. Putting equation 3 in 1 we get,

$$
\begin{align*}
& \mathrm{E}_{\mathrm{Ki}}=\frac{1}{2} \mathrm{~m}_{\mathrm{i}}\left[\left(\boldsymbol{v}_{\mathrm{i}}^{\prime}+\boldsymbol{V}_{\mathrm{cm}}\right) \cdot\left(\boldsymbol{v}_{\mathrm{i}}^{\prime}+\boldsymbol{V}_{\mathrm{cm}}\right)\right]=\frac{1}{2} \mathrm{~m}_{\mathrm{i}}\left[\left(\boldsymbol{v}_{i}^{2 \prime}+2 \boldsymbol{v}_{\mathrm{i}}^{\prime} \cdot \boldsymbol{V}_{\mathrm{cm}}+\boldsymbol{V}_{c m}^{2}\right)\right] \\
& \mathrm{E}_{\mathrm{Ki}}=\frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{v}_{i}^{\prime 2}+\mathrm{m}_{\mathrm{i}} \boldsymbol{v}_{\mathrm{i}}^{\prime} \cdot \boldsymbol{V}_{\mathrm{cm}}+\frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{V}_{c m}^{2} \tag{4}
\end{align*}
$$

Sum of Kinetic energy of all the particles can be obtained from equation 4

$$
\begin{align*}
& \mathrm{E}_{\mathrm{K}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{E}_{\mathrm{Ki}}=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left[\frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{v}_{i}^{\prime 2}+\mathrm{m}_{\mathrm{i}} \boldsymbol{v}_{\mathrm{i}}^{\prime} \cdot \boldsymbol{V}_{\mathrm{cm}}+\frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{V}_{c m}^{2}\right] \\
& \mathrm{E}_{\mathrm{K}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{v}_{i}^{\prime 2}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{v}_{\mathrm{i}}^{\prime} \cdot \boldsymbol{V}_{\mathrm{cm}}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{V}_{c m}^{2} \\
& \mathrm{E}_{\mathrm{K}}=\frac{1}{2} \boldsymbol{V}_{c m}^{2} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{v}_{i}^{\prime 2}+\boldsymbol{V}_{\mathrm{cm}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{v}_{\mathrm{i}}^{\prime} \\
& \mathrm{E}_{\mathrm{K}}=\frac{1}{2} \boldsymbol{V}_{c m}^{2} \mathrm{M}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{v}_{i}^{\prime 2}+\boldsymbol{V}_{\mathrm{cm}} \frac{\mathrm{~d}}{\mathrm{dt}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}^{\prime} \tag{5}
\end{align*}
$$

Now last term in above equation which is

$$
\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}^{\prime}=0
$$

would vanish as it defines the null vector because

$$
\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \boldsymbol{r}_{\mathrm{i}}^{\prime}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}\left(\boldsymbol{r}_{\mathrm{i}}-\boldsymbol{R}_{\mathrm{cm}}\right)=\mathrm{M} \boldsymbol{R}_{\mathrm{cm}}-\mathrm{M} \boldsymbol{R}_{\mathrm{cm}}=0
$$

Therefore kinetic energy of the system of particles is,

$$
\begin{equation*}
\mathrm{E}_{\mathrm{K}}=\frac{1}{2} \mathrm{MV}_{\mathrm{cm}}^{2}+\frac{1}{2} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \mathrm{v}_{\mathrm{i}}^{2}=\mathrm{E}_{\mathrm{Kcm}}+\mathrm{E}_{\mathrm{K}}^{\prime} \tag{6}
\end{equation*}
$$

where,
$\mathrm{E}_{\mathrm{Kcm}}=\frac{1}{2} \mathrm{~V}_{c m}^{2} \mathrm{M}$
is the kinetic energy obtained as if all the mass were concentrated at the centre of mass and
$\mathrm{E}_{\mathrm{K}}^{\prime}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{1}{2} \mathrm{~m}_{\mathrm{i}} \boldsymbol{v}_{i}^{\prime 2}$
is the kinetic energy of the system of particle w.r.t. the centre of mass.

Hence it is clear from equation 6 that kinetic energy of the system of particles consists of two parts: the kinetic energy obtained as if all the mass were concentrated at the centre of mass plus the kinetic energy of motion about the centre of mass.
If there were no external force acting on the particle system then the velocity of the centre of mass of the system will remain constant and Kinetic Energy of the system would also remain constant.

## (8) Two particle system and reduced mass

Two body problems with central forces can always be reduced to the form of one body problems.
Consider a system made up of two particles. For an observer in any inertial frame of refrence relative motion of these two particles can be represented by the motion of a fictitious particle.
The mass of this fictetious particle is known as the reduced mass of two particle system.
Consider a system of two particles of mass $m_{1}$ and $m_{2}$ respectively. Let $O$ be the origin of any inertial frame of refrance and $\mathbf{r}_{1}$ and $\mathbf{r}_{2}$ be the position vectors of these particles at any time $t$ w.r.t. origin $O$ as shown bellow in the figure.


Figure 5. Two particle system If no external force is acting on the system then the force acting on the system would be equal to mutual interaction between two particles. Let the force acting on $m_{1}$ due to $m_{2}$ be $F_{21}$ and force acting on $m_{2}$ due to $m_{1}$ be $F_{12}$ then equation of motion for particles $m_{1}$ and $m_{2}$ would be
$\boldsymbol{F}_{21}=\mathrm{m}_{1} \frac{\mathrm{~d}^{2} \boldsymbol{r}_{1}}{\mathrm{dt}^{2}}$
and,
$\boldsymbol{F}_{12}=\mathrm{m}_{2} \frac{\mathrm{~d}^{2} \boldsymbol{r}_{2}}{\mathrm{dt}^{2}}$
from 1 and 2
$\frac{\mathrm{d}^{2} \boldsymbol{r}_{1}}{\mathrm{dt}^{2}}=\frac{\boldsymbol{F}_{21}}{\mathrm{~m}_{1}}$
and
$\frac{\mathrm{d}^{2} \boldsymbol{r}_{2}}{\mathrm{dt}^{2}}=\frac{\boldsymbol{F}_{12}}{\mathrm{~m}_{2}}$
From the figure
$\boldsymbol{r}_{12}=\boldsymbol{r}_{2}-\boldsymbol{r}_{1}$
so,
$\frac{\mathrm{d}^{2} \boldsymbol{r}_{12}}{\mathrm{dt}^{2}}=\frac{\mathrm{d}^{2} \boldsymbol{r}_{2}}{\mathrm{dt}^{2}}-\frac{\mathrm{d}^{2} \boldsymbol{r}_{1}}{\mathrm{dt}^{2}}$
putting 3 and 4 in 6 we get
$\frac{\mathrm{d}^{2} \boldsymbol{r}_{12}}{\mathrm{dt}^{2}}=\frac{\boldsymbol{F}_{12}}{\mathrm{~m}_{2}}-\frac{\boldsymbol{F}_{21}}{\mathrm{~m}_{1}}$
but from Newton's first law of motion we have $F_{21}=-F_{12}$
then from equation 7 we have
$\frac{\mathrm{d}^{2} \boldsymbol{r}_{12}}{\mathrm{dt}^{2}}=\frac{\boldsymbol{F}_{12}}{\mathrm{~m}_{2}}+\frac{\boldsymbol{F}_{12}}{\mathrm{~m}_{1}}=\boldsymbol{F}_{12}\left(\frac{1}{\mathrm{~m}_{1}}+\frac{1}{\mathrm{~m}_{2}}\right)$
or, $\frac{\mathrm{d}^{2} \boldsymbol{r}_{12}}{\mathrm{dt}^{2}}=\boldsymbol{F}_{12}\left(\frac{\mathrm{~m}_{1}+\mathrm{m}_{2}}{\mathrm{~m}_{1} \mathrm{~m}_{2}}\right)$
$\boldsymbol{F}_{12}=\left(\frac{\mathrm{m}_{1} \mathrm{~m}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right) \frac{\mathrm{d}^{2} \boldsymbol{r}_{12}}{\mathrm{dt}^{2}}$
or, $\boldsymbol{F}_{12}=\mu \frac{\mathrm{d}^{2} \boldsymbol{r}_{12}}{\mathrm{dt}^{2}}$
where, $\mu=\left(\frac{\mathrm{m}_{1} \mathrm{~m}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right)$
is known as reduced mass of the system.
This equation 8 represents a one body problem, because it is similar to the equation of motion of single particle of mass $\mu$ at a vector distance $\mathbf{r}_{12}$ from one of thr two particles, considered as the fixed centre of force.
Thus original problem involving two particle system has now been reduced to that of one particle system which is easier to solve then original two body problem.
Case 1. $\mathrm{m}_{1} \ll \mathrm{~m}_{2}$
If the mass of any one particle in two particle system is very very less in comparison to other particle like in earth-satellite system then reduced mass of the system would be
$\mu=\left(\frac{\mathrm{m}_{1} \mathrm{~m}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right)=\left(\frac{\mathrm{m}_{1}}{1+\left(\mathrm{m}_{1} / \mathrm{m}_{2}\right)}\right)$
or, $\mu \approx\left(1-\frac{\mathrm{m}_{1}}{\mathrm{~m}_{2}}\right) \mathrm{m}_{1} \quad$ (using binomial theorem)
or, $\mu \approx \mathrm{m}_{1}$
So the reduced mass of the two particle system would be equal to the particle having lesser mass.
Case 2. $m_{1}=m_{2}=m$
If the masses 08 the particles of a two particle system are same then
$\mu=\left(\frac{\mathrm{m}_{1} \mathrm{~m}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right)=\frac{\mathrm{m}^{2}}{2 \mathrm{~m}}=\frac{\mathrm{m}}{2}$
Hence reduced mass of the system would be equal to the one half of the mass of a single particle.
(9) Linear momentum and principle of conservation of linear

## momentum

Product of mass and velocity of any particle is defined as the linear momentum of the particle. It is a vector quantity and its direction is same as the direction of velocity of the particle.
Linear momentum is represented by $\mathbf{p}$. If $m$ is the mass of the particle moving with velocity $\mathbf{v}$ then linear momentum of the particle would be
$\mathrm{p}=\mathrm{mv}$
like $\mathbf{v}, \mathbf{p}$ also depends on the frame of refrance of the observer.
If in a many particle system $m_{1}, m_{2}, m_{3}, \ldots \ldots \ldots \ldots, m_{n}$ are the masses and $\mathbf{v}_{1}, v_{2}, v_{3}, \ldots \ldots \ldots$. $\ldots, \mathbf{v}_{\mathrm{n}}$ are the velocities of the respective particles then total linear momentum of the system would be

$$
\begin{aligned}
& \mathbf{p}=\mathbf{p}_{1}+\mathbf{p}_{2}+\mathbf{p}_{3}+\ldots \ldots \ldots . . \mathbf{p}_{\mathrm{n}} \\
& \mathbf{p}=m_{1} \mathbf{v}_{\mathbf{1}}+m_{2} \mathbf{v}_{2}+m_{3} \mathbf{v}_{3}+\ldots \ldots \ldots .+m_{n} \mathbf{v}_{\mathbf{n}} \\
& \mathbf{p}=M \mathbf{V}_{\mathrm{cm}}
\end{aligned}
$$

where $M$ is the total mass of the system and $V_{c m}$ is the velocity of centre of mass of the system
Hence from equation 2 we came to know that total linear momentum of a many particle system is equal to the product of the total mass of the system and velocity of centre of mass of the system.

Differentiating equation 2 w.r.t. t we get

$$
\begin{equation*}
\frac{\mathrm{d} \boldsymbol{p}}{\mathrm{dt}}=\mathrm{M} \frac{\mathrm{~d} \boldsymbol{V}_{\mathrm{cm}}}{\mathrm{dt}}=\mathrm{M} \boldsymbol{a}_{\mathrm{cm}} \tag{3}
\end{equation*}
$$

but, $M a_{\mathrm{cm}}=\mathrm{F}_{\text {ext }}$ which is the external force acting on the system. Therefore,

$$
\begin{equation*}
\frac{\mathrm{d} \boldsymbol{p}}{\mathrm{dt}}=\boldsymbol{F}_{\mathrm{ext}} \tag{4}
\end{equation*}
$$

like this the rate of change of momentum of a many particle system comes out to be equal to the resultant external force acting on the particle.
If external force acting on the system is zero then,

```
\(\frac{\mathrm{d} p}{\mathrm{dt}}=0\)
or, \(\boldsymbol{p}=\) constan t
or, \(p_{1}+p_{2}+p_{3}+\ldots \ldots . .+p_{1}=\) const.
\(\mathrm{M} \boldsymbol{V}_{\mathrm{cm}}=\boldsymbol{p}=\mathrm{const}\).
or, \(\boldsymbol{V}_{\mathrm{cm}}=\) const.
```

Hence we conclude that when resultant external force acting on any particle is zero then total linear momentum of the system remains constant. This is known as law of conservation of linear momentum.
Above equation 5 is equivalent to following scalar quantities
$\mathrm{p}_{\mathrm{x} 1}+\mathrm{p}_{\mathrm{x} 2}+\mathrm{p}_{\mathrm{x} 3}+\ldots \ldots \ldots . . \mathrm{p}_{\mathrm{xn}}=$ const.
$\mathrm{p}_{\mathrm{y} 1}+\mathrm{p}_{\mathrm{y} 2}+\mathrm{p}_{\mathrm{y} 3}+\ldots \ldots \ldots .+\mathrm{p}_{\mathrm{yn}}=$ const.
$\mathrm{p}_{\mathrm{z} 1}+\mathrm{p}_{\mathrm{z} 2}+\mathrm{p}_{\mathrm{z} 3}+\ldots \ldots \ldots .+\mathrm{p}_{\mathrm{zn}}=$ const.
Equation 6 shows the total linear momentum of the system in terms of $x, y$ and $z$ co-ordinates and also shows that they remain constant or conserved in absence of any externally applied force.
The law of conservation of linear momentum is the fundamental and exact law of nature. No violation of it has ever been found. This law has been established on the basis of Newton's law but this law holds true in the situations where Newtonian mechanics fails.

## (10) Centre of mass frame of refrance

If we attach an inertial frame of refrance with the centre of mass of many particle system then centre of mass in that frame of refrance would be at rest or, $\mathbf{V}_{\mathrm{cm}}=0$, and such type of refrance frames are known as centre of mass frame of refrance.
Total linear momentum of a many particle system is zero in centre of mass frame of refrance i.e., $\mathbf{p}_{\mathrm{cm}}=\mathrm{M} \mathbf{V}_{\mathrm{cm}}=0$ since $\mathbf{V}_{\mathrm{cm}}=0$.
Therefore C-refrance frames are also known as zero momentum refrance frames.
Since in absence of any external force the centre of mass of any system moves with constant velocity in inertial frame of refrance therefore for a many particle system C-rame of refrance is an inertial frame of refrence.
Refrance frames connected to laboratory are known as L-frame of refrance or lebiratory frame of refrance.

## (11) Collisions

Collision between two particles is defined as mutual interaction between particles for a short interval of time as a result of which energy and momentum of particle changes.
Collision between two billiard balls or between two automobiles on road are few examples of collisions from our everyday life. Even gas atoms and molecules at room temperature keep on colliding against each other. For the collision to take place, physical contact is not necessary. In cas of Rutherford alpha scattering experiment , the alpha particles are scattered due to electrostatic interaction between the alpha particles and the nucleus from a distance i.e., no physical contact occurs between the alpha particles and the nucleus. Thus, in physics collision is said to have occured, if two particles physically collide with each other or even when the path of motion of one particle is affected by other.
In the collision of two particles law of conservation of momentum always holds true but in some collisions Kinetic energy is not always conserved.
Hence collisions are of two types on the basis of conservation of energy.
(i) Perfectly elastic collision

Those collisions in which both momentum and kinetic energy of system are conserved are called elastic collisions for example elastic collision occurs between the molecules of a gas. This type of collision mostly takes place between the atoms, electrons and protons.

## Characterstics of elastic collision

(a) Total momentum is conserved.
(b) Total energy is conserved.
(c) Total kinetic energy is conserved.
(d) Total mechanical energy is not converted into any other form of energy.
(e) Forces involved during interaction are conservative in nature.

Consider two particles whose masses are $m_{1}$ and $m_{2}$ respectively and they collide each other with velocity $\mathbf{u}_{1}$ and $\mathbf{u}_{2}$ and after collision their velocities become $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$ respectively.
If collision between these two particles is elastic one then from law of conservation of momentum we have
$m_{1} \mathbf{u}_{1}+\mathrm{m}_{2} \mathbf{u}_{2}=\mathrm{m}_{1} \mathbf{v}_{1}+\mathrm{m}_{2} \mathbf{v}_{2}$
and from the law of conservation of energy we have
$\frac{1}{2} m_{1} \mathbf{u}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{u}_{2}^{2}=\frac{1}{2} m_{1} \mathbf{v}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{v}_{2}^{2}$

Those collisions in which momentum of system is conserved but kinetic energy of the system is not conserved are known as inelastic collision.
Here in inelastic collision two bodies stick to each other after collision as a bullet hit its target and remain embedded in the target.
In this case some of the kinetic energy is converted into heat or is used up in in doing work in deforming bodies for example when two cars collide their metal parts are bet out of shape.
Characterstics of inelastic collision
(a) Total momentum is conserved.
(b) Total energy is conserved.
(c) Total kinetic energy is not conserved.
(d) A part or whole of whole mechanical energy may be converted into other forms of energy.
(e) Some or all forces involved during interaction are non-conservative in nature.

Consider two particles whose masses are $\mathrm{m}_{1}$ and $\mathrm{m}_{2}$ respectively and they collide each other with velocity $\mathbf{u}_{1}$
and $\mathbf{u}_{2}$ respectively.
If the collision between these two particles is inelastic then these two particles would stick to each other and after collision they move with velocity $\mathbf{v}$ then from law of conservation of momentum we have

$$
\begin{aligned}
& m_{1} \boldsymbol{u}_{1}+m_{2} \boldsymbol{u}_{2}=\left(m_{1}+m_{2}\right) \boldsymbol{v} \\
& \boldsymbol{v}=\frac{m_{1} \boldsymbol{u}_{1}+m_{2} \boldsymbol{u}_{2}}{\left(m_{1}+m_{2}\right)}
\end{aligned}
$$

Kinetic energy of particles before collisions is
$K \cdot E_{\cdot_{i}}=\frac{1}{2} m_{1} \mathbf{u}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{u}_{2}^{2}$
and kinetic energy of particles after collisions is
$K \cdot E_{\cdot f}=\frac{1}{2}\left(m_{1}+m_{2}\right) \mathbf{v}^{2}$
by law of conserevation of energy
$\frac{1}{2} m_{1} \mathbf{u}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{u}_{2}^{2}=\frac{1}{2}\left(m_{1}+m_{2}\right) \mathbf{v}^{2}+Q$
where $Q$ is the loss in kinetic energy of particles during collision.

## (12) Head on elastic collision of two particles in L-frame of refrance

Consider two particles whose masses are $m_{1}$ and $m_{2}$ respectively and they collide each other with velocity $\mathbf{u}_{1}$ and $\mathbf{u}_{2}$ and after collision their velocities become $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$ respectively.
Collision between these two particles is head on elastic collision. From law of conservation of momentum we have
$\mathrm{m}_{1} \mathbf{u}_{1}+\mathrm{m}_{2} \mathbf{u}_{2}=\mathrm{m}_{1} \mathbf{v}_{1}+\mathrm{m}_{2} \mathbf{v}_{2}$
and from law of conservation of kinetic energy for elastic collision we have
$\frac{1}{2} m_{1} \mathbf{u}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{u}_{2}^{2}=\frac{1}{2} m_{1} \mathbf{v}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{v}_{2}^{2}$
rearranging equation 1 and 2 we get
$m_{1}\left(u_{1}-\mathbf{v}_{1}\right)=m_{2}\left(\mathbf{v}_{2}-\mathbf{u}_{2}\right)$
and
$m_{1}\left(\mathbf{u}_{1}^{2}-\mathbf{v}_{1}^{2}\right)=m_{2}\left(\mathbf{v}_{2}^{2}-\mathbf{u}_{2}^{2}\right)$
dividing equation 4 by 3 we get

$$
u_{1}+v_{1}=u_{2}+v_{2}
$$

$$
\begin{equation*}
\mathbf{u}_{2}-\mathbf{u}_{1}=-\left(\mathbf{v}_{2}-\mathbf{v}_{1}\right) \tag{5}
\end{equation*}
$$

where $\left(\mathbf{u}_{2}-\mathbf{u}_{1}\right)$ is the relative velocity of second particle w.r.t. first particle before collision and $\left(\mathbf{v}_{2}-\mathbf{v}_{1}\right)$ is the relative velocity of second particle w.r.t. first after collision.
From equation 5 we come to know taht in a perfectly elastic collision the magnitude of relative velocity remain unchanged but its direction is reversed. With the help of above equations we can find the values of $\mathbf{v}_{2}$ and $\mathbf{v}_{1}$, so from equation 5

$$
\begin{align*}
& \mathbf{v}_{1}=\mathbf{v}_{2}-u_{1}+u_{2}  \tag{6}\\
& \mathbf{v}_{2}=v_{1}+u_{1}-u_{2} \tag{7}
\end{align*}
$$

Now putting the value of $\mathbf{v}_{1}$ from equation 6 in equation 3 we get
$\mathrm{m}_{1}\left(\mathbf{u}_{1}-\mathbf{v}_{2}+\mathbf{u}_{1}-\mathbf{u}_{2}\right)=\mathrm{m}_{2}\left(\mathbf{v}_{2}-\mathbf{u}_{2}\right)$
On solving the above equation we get value of $\mathbf{v}_{2}$ as
$\mathbf{v}_{2}=\left(\frac{2 m_{1}}{m_{1}+m_{2}}\right) \mathbf{u}_{1}+\left(\frac{m_{2}-m_{1}}{m_{1}+m_{2}}\right) \mathbf{u}_{2}$
Similarly putting the value of $\mathbf{v}_{2}$ from equation 7 in equation 3 we get
$\mathbf{v}_{1}=\left(\frac{2 m_{2}}{m_{1}+m_{2}}\right) \mathbf{u}_{2}+\left(\frac{m_{1}-m_{2}}{m_{1}+m_{2}}\right) \mathbf{u}_{1}$

Total kinetic energy of particles before collision is
$K E_{i}=\frac{1}{2} m_{1} \mathbf{u}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{u}_{2}^{2}$
and total K.E. of particles after collision is
$K E_{f}=\frac{1}{2} m_{1} \mathbf{v}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{v}_{2}^{2}$
Ratio of initial and final K.E. is
$\frac{K E_{i}}{K E_{f}}=\frac{\frac{1}{2} m_{1} \mathbf{u}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{u}_{2}^{2}}{\frac{1}{2} m_{1} \mathbf{v}_{1}^{2}+\frac{1}{2} m_{2} \mathbf{v}_{2}^{2}}=1$

## Special cases

Case I: When the mass of both the particles are equal i.e., $m_{1}=m_{2}$ then from equation 8 and $9, \mathbf{v}_{2}=\mathbf{u}_{1}$ and $\mathbf{v}_{1}=\mathbf{u}_{2}$. Thus if two bodies of equal masses suffer head on elastic collision then the particles will exchange their velocities. Exchange of momentum between two particles suffering head on elastic collision is maximum when mass of both the particles is same.
Case II: when the target particle is at rest $i, e u_{2}=0$
From equation (8) and (9)
$v_{2}=\left(\frac{2 m_{1}}{m_{1}+m_{2}}\right) u_{1}$
$v_{1}=\left(\frac{m_{1}-m_{2}}{m_{1}+m_{2}}\right) u_{1}$
Hence some part of the KE which is transformed into second particle would be
$\frac{\frac{1}{2} m_{2} v_{2}^{2}}{\frac{1}{2} m_{1} u_{1}^{2}}=\frac{\frac{1}{2} m_{2}\left(\frac{2 m_{1} u_{1}}{m_{1}+m_{2}}\right)^{2}}{\frac{1}{2} m_{1} u_{1}^{2}}$
$=\frac{4 \mathrm{~m}_{1} \mathrm{~m}_{2}}{\left(\mathrm{~m}_{1}+\mathrm{m}_{2}\right)^{2}}=\frac{4 \frac{\mathrm{~m}_{2}}{\mathrm{~m}_{1}}}{\left(1+\frac{\mathrm{m}_{2}}{\mathrm{~m}_{1}}\right)^{2}}$
when $m_{1}=m_{2}$, then in this condition $v_{0}=0$ and $v_{2}=u_{1}$ and part of the KE transferred would be $=1$
Therefore after collisom first particle moving with initial velocity $u_{1}$ would come to rest and the second particle which was at rest would start moving with the velocity of first particle. Hence in this case when $m_{1}=m_{2}$ transfer of energy is $100 \%$.if $m_{1}>m_{2}$ or
$m_{1}<m_{2}$, then energy transformation is not $100 \%$

## Case III:

if $m_{2} \ggg>m_{1}$ and $u_{2}=0$ then from equation (10) and (11)
$v_{1} \square-u_{1}$ and $v_{2}=0$
For example when a ball thrown upwards collide with earth

## Case IV:

if $m_{1} \ggg>m_{2}$ and $u_{2}=0$ then from equation (10) and (11)
$v_{1} \square u_{1}$ and $v_{2}=2 u_{1}$
Therefore when a heavy particle collide with a very light particle at rest ,then the heavy particle keeps on moving with the same velocity and the light particle come in motion with a velocity double that of heavy particle

## (13) Head on collison of two particles in C frame of refrence

Consider two particles of mass $m_{1}$ and $m_{2}$ having position vectors $\mathbf{r}_{1}$ and $\mathbf{r}_{2}$ respectively And position vector of the CEnter of mass of the system would be $\mathbf{R}_{\mathbf{c m}}$
then
$\mathbf{R}_{\mathrm{cm}}=\frac{m_{1} \mathbf{r}_{1}+m_{2} \mathbf{r}_{2}}{m_{1}+m_{2}}$
Velocity of the center of mass would be
$\boldsymbol{v}_{c m}=\frac{\mathrm{d} \boldsymbol{R}_{c m}}{\boldsymbol{d t}}=\frac{\mathrm{m}_{1} \frac{\mathrm{~d} \boldsymbol{r}_{1}}{\mathrm{dt}}+\mathrm{m}_{2} \frac{\mathrm{~d} \boldsymbol{r}_{2}}{\mathrm{dt}}}{\mathrm{m}_{1}+\mathrm{m}_{2}}=\frac{\mathrm{m}_{1} \boldsymbol{u}_{1}+\mathrm{m}_{2} \boldsymbol{u}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}$
Intial velocity of the $m_{1}$ w.r.t center of mass frame of refrence is

$$
\begin{align*}
\boldsymbol{u}_{1}^{\prime} & =\boldsymbol{u}_{1}-\boldsymbol{v}_{c m}=\boldsymbol{u}_{1}-\left(\frac{\mathrm{m}_{1} \boldsymbol{u}_{1}+\mathrm{m}_{2} \boldsymbol{u}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right) \\
& =\frac{\mathrm{m}_{1} \boldsymbol{u}_{1}+\mathrm{m}_{2} \boldsymbol{u}_{1}-\mathrm{m}_{1} \boldsymbol{u}_{1}-\mathrm{m}_{2} \boldsymbol{u}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}=\frac{\mathrm{m}_{2} \boldsymbol{u}_{1}-\mathrm{m}_{2} \boldsymbol{u}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}} \tag{17}
\end{align*}
$$

Similarly Intial velocity of $m_{2}$ w.r.t center of mass frame of refrence is
$\boldsymbol{u}_{2}^{\prime}=\boldsymbol{u}_{2}-\boldsymbol{v}_{c m}=\boldsymbol{u}_{2}-\left(\frac{\mathrm{m}_{1} \boldsymbol{u}_{1}+\mathrm{m}_{2} \boldsymbol{u}_{2}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right)=\frac{\mathrm{m}_{1} \boldsymbol{u}_{2}-\mathrm{m}_{1} \boldsymbol{u}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}$
Total linear momentum before collison in absence of external force in $C$ frame of refrence would be
$=m_{1} \mathbf{u}_{1}{ }^{\prime}+\mathrm{m}_{2} \mathbf{u}_{\mathbf{2}}{ }^{\prime}$
$=0$
So $\mathbf{u}_{\mathbf{2}}{ }^{\prime}=\left(\mathrm{m}_{1} / \mathrm{m}_{2}\right) \mathbf{u}_{\mathbf{1}}{ }^{\prime}$
If $\mathbf{v}_{\mathbf{1}}{ }^{\prime}$ and $\mathbf{v}_{\mathbf{2}}{ }^{\prime}$ are the velocites of mass $\mathrm{m}_{1}$ and $\mathrm{m}_{2}$ respectively after collision then by law of conservation of linear momentum
$m_{1} \mathbf{v}_{\mathbf{1}}{ }^{\prime}+\mathrm{m}_{2} \mathbf{v}_{\mathbf{2}}{ }^{\prime}=0$
$\mathbf{v}_{\mathbf{2}}=\left(\mathrm{m}_{1} / \mathrm{m}_{2}\right) \mathbf{v}_{\mathbf{1}}$
Since the collision is elastic,Kinetic energy will be conserved
$\frac{1}{2} m_{1} u_{1}^{\prime 2}+\frac{1}{2} m_{2} u_{2}^{\prime 2}=\frac{1}{2} m_{1} v_{1}^{\prime 2}+\frac{1}{2} m_{2} v_{2}^{\prime 2}$
Putting the values of $u_{2}^{\prime}$ and $v_{2}^{\prime}$, we get
$m_{1} u_{1}^{\prime 2}+m_{2}\left(\frac{m_{1} u_{1}^{\prime}}{m_{2}}\right)^{2}=m_{1} v_{1}^{\prime 2}+m_{2}\left(\frac{m_{1} v_{1}^{\prime}}{m_{2}}\right)^{2}$
hence $v_{1}^{\prime 2}=u_{1}^{\prime 2}$
From which $\left|\mathbf{v}_{\mathbf{1}}{ }^{\prime}\right|=\left|\mathbf{u}_{\mathbf{1}}{ }^{\prime}\right|$ and $\left|\mathbf{v}_{\mathbf{2}}{ }^{\prime}\right|=\left|\mathbf{u}_{\mathbf{2}}{ }^{\prime}\right|$
hence after collison velocities of particles remain unchanged in center of mass frame of refrence.If the collision is one dimmension then because of the collsion direction of these would be opposite to that of their intial velocites

$$
\begin{align*}
& \mathbf{v}_{1}^{\prime}=-\mathbf{u}_{1}^{\prime}=-\left(\frac{m_{2} \mathbf{u}_{1}-m_{2} \mathbf{u}_{2}}{m_{1}+m_{2}}\right)  \tag{19}\\
& \mathbf{v}_{2}^{\prime}=-\mathbf{u}_{2}^{\prime}=-\left(\frac{m_{1} \mathbf{u}_{2}-m_{1} \mathbf{u}_{1}}{m_{1}+m_{2}}\right) \tag{20}
\end{align*}
$$

## (14) Head on inelastic collision of two particles which stick together

## A) In laboraratory frame of refrence

Consider two particles whose masses are $m_{1}$ and $m_{2}$. Let $\mathbf{u}_{1}$ and $\mathbf{u}_{2}$ be the respective velocities before collision
Let both the particles stick together after collision and moves with the same velocity $\mathbf{v}$. Then from law of conservation of linear momentum
$m_{1} \mathbf{u}_{\mathbf{1}}+\mathrm{m}_{2} \mathbf{u}_{2}=\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right) \mathbf{v}$
or
$\mathbf{v}=\frac{m_{1} \mathbf{u}_{1}+m_{2} \mathbf{u}_{2}}{m_{1}+m_{2}}$
If we consider second particle to be stationary or at rest then $\mathbf{u}_{\mathbf{2}}=0$
then
$\mathrm{m}_{1} \mathbf{u}_{\mathbf{1}}=\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right)$
or
$\mathbf{v}=\frac{m_{1} \mathbf{u}_{1}}{m_{1}+m_{2}}$
Hence $|\mathbf{v}|<\left|\mathbf{u}_{\mathbf{1}}\right|$
Kinetic energy before collision is
$K E_{1}=(1 / 2) m_{1} u_{1}{ }^{2}$
After collison KE of the system is
$\mathrm{KE}_{2}=(1 / 2)\left(\mathrm{m}_{1}+\mathrm{m}_{2}\right) \mathrm{v}^{2}$
$\mathrm{KE}_{2}=\frac{1}{2}\left(\mathrm{~m}_{1}+\mathrm{m}_{2}\right)\left(\frac{\mathrm{m}_{1} \mathrm{u}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right)^{2}=\frac{1}{2} \frac{\mathrm{~m}_{1}^{2} \mathrm{u}_{1}^{2}}{\left(\mathrm{~m}_{1}+\mathrm{m}_{2}\right)}$
hence
$\frac{\mathrm{K}_{2}}{\mathrm{~K}_{1}}=\frac{\frac{1}{2} \frac{\mathrm{~m}_{1}^{2} \mathrm{u}_{1}^{2}}{\left(\mathrm{~m}_{1}+\mathrm{m}_{2}\right)}}{\frac{1}{2} \mathrm{~m}_{1} \mathrm{u}_{1}^{2}}=\frac{\mathrm{m}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}<1$
Hence from equation we come to know that $\mathrm{K}_{2}<\mathrm{K}_{1}$ hence energy loss would be there after thye collision of the particles

Velocity of center of mass is
$\mathbf{v}_{\mathrm{cm}}=\frac{m_{1} \mathbf{u}_{1}}{m_{1}+m_{2}}$
when second particle is at rest.
It is clear that Center of masss frame of refrence moves with velocity $v_{c m}$ w.r.t laboratory frame of refrence.
Hence velocity of particle having mass $m_{1}$ in $C$-frame of refrence is
$\boldsymbol{u}_{1}^{\prime}=\boldsymbol{u}_{1}-\boldsymbol{v}_{c m}=\boldsymbol{u}_{1}-\left(\frac{\mathrm{m}_{1} \boldsymbol{u}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right)=\frac{\mathrm{m}_{2} \boldsymbol{u}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}$
And velocity of particle having mass $m_{2}$ in C-frame of refrence would be
$\boldsymbol{u}_{2}^{\prime}=\boldsymbol{0}-\boldsymbol{v}_{c m}=\boldsymbol{0}-\left(\frac{\mathrm{m}_{1} \boldsymbol{u}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right)$
let $v$ ' be the joint velocity of the particles which stick to each other after the collision in center of mass frame of refrence .So by the law of conservation of linear momentum
$\left(m_{1}+m_{2}\right) \mathbf{v}^{\prime}=m_{1} \mathbf{u}_{1}{ }^{\prime}+m_{2} \mathbf{u}_{2}{ }^{\prime}$
Substituting the values of $\mathbf{u}_{1}{ }^{\prime}$ and $\mathbf{u}_{2}$ ' we find
$\left(m_{1}+m_{2}\right) \mathbf{v}^{\prime}=0$
but $\left(m_{1}+m_{2}\right)$ can be zero so
$\mathbf{v}^{\prime}=0$
Therefore after collision total momentum of the particles comes out to be zero.Because both the particles stick to each other after collision so the velocity of joint particle mass i.e $\left(m_{1}+m_{2}\right)$ would be zero. Therefore in center of mass frame of refrence ,particle which stick together will remain stationary.

## (15) Deflection of an moving particle by a particle at rest during

## perfectly elastic collision or elastic collision in two dimension

## A) In labaoratory frame of refrence

Let $m_{1}$ and $m_{2}$ be the two mass particle in a laboratory frame of refrence and $m_{1}$ collide with $m_{2}$ which is initailly is at rest.Let the velocity of mass $m_{1}$ before collison be $u_{1}$ and after the collison it moves with a velocity $\mathbf{v}_{\mathbf{1}}$ and is delfected by the angle $\theta_{1}$ withs its incident direction and $\mathrm{m}_{2}$ after the collision moves with the velocity $\mathbf{v}_{\mathbf{2}}$ and it is deflected by an angle $\theta_{2}$ with its incident direction


Figure 6. Elastic collision in two dimension in laboratory frame of refrance
From law of conservation of linear momentum ,for components along x-axis
$m_{1} u_{1}=m_{1} v_{1} \cos \theta_{1}+m_{2} v_{2} \cos \theta_{2}--(1)$
For components along y-axis
$0=m_{1} v_{1} \sin \theta_{1}-m_{2} v_{2} \sin \theta_{2}--(2)$
And from law of conservation of energy
$(1 / 2) m_{1} u_{1}^{2}=(1 / 2) m_{1} v_{1}^{2}+(1 / 2) m_{2} v_{2}^{2}---(3)$
Analysing above equations we come to know that we have to find values of four unknown quantities $\mathrm{v}_{1}, \mathrm{v}_{2}, \theta_{1}, \theta_{2}$ with the help of above three equations which is impossible as we need to have atleast four equations for finding out the values of four unknown quantities. Hence this problem can be solved in C frame of refrence
B) In center of mass frame of refrence

Velocity of the center of mass wrt to $L$-frame of refrence is equation
$\mathbf{v}_{\mathbf{c m}}=\frac{m_{1} \mathbf{u}_{1}}{m_{1}+m_{2}}$
In C frame of refrence ,the center of mass remain stationary ,the velocity of mass $m_{1}$ before collision w.r.t Cframe of refrence is
$\mathbf{u}_{\mathbf{1}}{ }^{\prime}=\mathbf{u}_{\mathbf{1}}-\mathbf{v}_{\mathbf{c m}}$
And that of mass $m_{2}$ is
$\mathbf{u}^{\prime}=0-\mathbf{v}=-\mathbf{v}$

After the collision velocites of $m_{1}$ and $m_{2}$ in C-frame of refrence would be

$$
\begin{align*}
& \mathbf{v}_{1}{ }^{\prime}=\mathbf{v}_{1}-v_{c m}  \tag{7}\\
& \mathbf{v}_{2}^{\prime}=\mathbf{v}_{2}-v_{c m} \tag{8}
\end{align*}
$$

Since center of mass remains stationary in C-frame of refrence hence total momentum would be zero. Therefore the momentum of both the massed would be equal and in opposite direction
So
$m_{1} \mathbf{u}_{1}{ }^{\prime}=-m_{2} \mathbf{u}_{2}{ }^{\prime}$
$m_{1} \mathbf{v}_{\mathbf{1}}{ }^{\prime}=-\mathrm{m}_{2} \mathbf{v}_{\mathbf{2}}{ }^{\prime}$

The above equation can be proved like this
$\mathrm{m}_{1} \mathbf{u}_{\mathbf{1}}{ }^{\prime}=\mathrm{m}_{1}\left(\mathbf{u}_{\mathbf{1}}-\mathbf{v}_{\mathbf{c m}}\right)$
$\mathrm{m}_{1} \boldsymbol{u}_{1}^{\prime}=\mathrm{m}_{1}\left[\boldsymbol{u}_{1}-\left(\frac{\mathrm{m}_{1} \boldsymbol{u}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}\right)\right]=\frac{\mathrm{m}_{1} \mathrm{~m}_{2} \boldsymbol{u}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}$
and $-\mathrm{m}_{2} \mathbf{u}_{\mathbf{2}}{ }^{\prime}=\mathrm{m}_{2} \mathbf{v}_{\mathbf{c m}}$
$-\mathrm{m}_{2} \boldsymbol{u}_{2}^{\prime}=\mathrm{m}_{2} \boldsymbol{v}_{c m}=\frac{\mathrm{m}_{1} \mathrm{~m}_{2} \boldsymbol{u}_{1}}{\mathrm{~m}_{1}+\mathrm{m}_{2}}$

From the above equations
$m_{1} \mathbf{u}_{\mathbf{1}}{ }^{\prime}=-\mathrm{m}_{2} \mathbf{u}_{\mathbf{2}}{ }^{\prime}$
Similarly we can prove that
$m_{1} \mathbf{v}_{\mathbf{1}}{ }^{\prime}=-\mathrm{m}_{2} \mathbf{v}_{\mathbf{2}}{ }^{\prime}$

It is clear from the above equations that after collisions the velocity of the particles i.e $\mathbf{v}_{\mathbf{1}}{ }^{\prime}, \mathbf{v}_{\mathbf{2}}{ }^{\prime}$ would be in opposite direction to each other and make same angles with the direction of the intial velocities of the particles as shown fig: (b)


Figure 7a. Before collision
From equation (9) and (10)
$\mathbf{u}_{\mathbf{2}}{ }^{\prime}=\left(\mathrm{m}_{1} / \mathrm{m}_{2}\right) \mathbf{u}_{\mathbf{1}}{ }^{\prime}$
and $\mathbf{v}_{\mathbf{2}}{ }^{\prime}=\left(\mathrm{m}_{1} / \mathrm{m}_{2}\right) \mathrm{b}>\mathrm{v}_{1}{ }^{\prime}$


Figure 7b. After collision

From the law of conservation of energy in this frame of refrence

$$
\begin{equation*}
\frac{1}{2} m_{1} u_{1}^{\prime 2}+\frac{1}{2} m_{2} u_{2}^{\prime 2}=\frac{1}{2} m_{1} v_{1}^{\prime 2}+\frac{1}{2} m_{2} v_{2}^{\prime 2} \tag{12}
\end{equation*}
$$

Putting the values of $u_{2}^{\prime}$ and $v_{2}^{*}$, we get
$m_{1} u_{1}^{\prime 2}+m_{2}\left(\frac{m_{1} u_{1}^{\prime}}{m_{2}}\right)^{2}=m_{1} v_{1}^{\prime 2}+m_{2}\left(\frac{m_{1} v_{1}^{\prime}}{m_{2}}\right)^{2}$
hence $v_{1}^{\prime 2}=u_{1}^{\prime 2}$
From which $\left|\mathbf{v}_{\mathbf{1}}{ }^{\prime}\right|=\left|\mathbf{u}_{\mathbf{1}}{ }^{\prime}\right|$ and $\left|\mathbf{v}_{\mathbf{2}}{ }^{\prime}\right|=\left|\mathbf{u}_{\mathbf{2}}{ }^{\prime}\right|$.
From these equations we came to know that magnitude of the velocities in C-frame of refrance, when the collision is elastic, does not change but their direction could change after collision.

## Rotation

## (1) Introduction

In this chapter, we will study about the special kind of motion of a system of particles that is rotation We see examples of rotational motion in our everyday life for example rotation of earth about its own axis create the cycle of day and night .Motion of wheel, gears ,motors, planet ,blades of the helicopter etc are all the example of rotational motion
To understand the rotational motion as a whole we are first required to understand the concept of angular position, velocity, acceleration, centripetal acceleration
Till now in our study of dynamics we have always analyzed motion of an object by considering it as a particle even when the size of the object is not negligible
In this process we represent object under consideration as a point mass and shape and size of the object remains irrelevant while discussing the particular problem under consideration
But this point mass or point particle model is inadequate for problems involving rigid body motion i.e. rigid body undergoing both translational and rotational motion
As an example consider the motion of a wheel, we cannot consider a wheel as a single particle because different parts of the wheel in motion has different velocities and acceleration
Here in this chapter we will consider rigid bodies having definite shape and size and are capable of having both rotational and translational motion

The rigid body is a body with a perfectly defined and unchanging shape that is no matter how the body moves, the distance between any two particles within the body remains constant Although the way we define rigid body gives us the definition of an idealized rigid body and real materials always deforms on the application of force and this idealized rigid body assumption can be used freely for the substances where deformation is negligibly small and can be neglected Motion of a rigid body in general can be considered to consist of a translation of center of mass of the body plus rotation of the body about an axis through the center of mass as shown below in the figure
 about center of mass

In this chapter we will concentrate on the simplest kind of rotation that is the rotation of the rigid body about the fixed axis

## (2) Angular velocity and angular acceleration

Consider a rigid body of arbitrary shape rotating about a fixed axis through point O and perpendicular to the plane of the paper as shown below in the figure-1 FIG
while the body is rotating each and every point in the body moves in a circle with their center lying on the axis of rotation and every point moves through the same angle during a particular interval of time Consider the position of a particle say ith particle at point $P$ at a distance $r_{i}$ from point $O$ and at an angle $\theta_{i}$ which OP makes with some reference line fixed in space say OX as shown below in the figure


Figure 2. A rigid body rotating about fixed axis through point O

If particle moves an small distance $\mathrm{ds}_{\mathrm{i}}$ along the arc of the circle in small amount of time dt then
$\mathrm{ds}_{\mathrm{i}}=\mathrm{v}_{\mathrm{i}} \mathrm{dt}----(1)$
where $v_{i}$ is the speed of the particle
$d \theta$ is the angle subtended by an arc of length $d s_{i}$ on the circumference of a circle of radius $r_{i}, S o d \theta$ ( in radians) would be equal to the length of the arc divided by the radius
i.e.
$\mathrm{d} \theta=\mathrm{ds} / \mathrm{r}_{\mathrm{i}}=\mathrm{v}_{\mathrm{i}} \mathrm{dt} / \mathrm{r}_{\mathrm{i}}---$-(2)
distance $\mathrm{ds}_{\mathrm{i}}$ would vary from particle to particle but angle $\mathrm{d} \theta$ swept out in a given time remain same for all the particles i.e. if particle at point P moves through complete circle such that
$\mathrm{d} \theta=2 \pi \mathrm{rad}$
Then all the other particles of the rigid body moves through the angular displacement $\mathrm{d} \theta=2 \pi$
So rate of change of angle w.r.t time i.e. $\mathrm{d} \theta / \mathrm{dt}$ is same for all particles of the rigid body and $\mathrm{d} \theta / \mathrm{dt}$ is known as angular velocity $\omega$ of the rigid body so
$\omega=\mathrm{d} \theta / \mathrm{dt}$
Putting equation (3) in equation (2) we find
$\mathrm{v}_{\mathrm{i}}=\mathrm{r}_{\mathrm{i}}(\mathrm{d} \theta / \mathrm{dt})=\mathrm{r}_{\mathrm{i}} \omega$---(4)
This shows that velocity of ith particle of the rigid body is related to its radius and the angular velocity of the rigid body
Angular velocity of a rotating rigid body can either be positive or negative. It is positive when the body is rotating in anticlockwise direction and negative when the body id rotating in clockwise direction Unit of angular velocity radian per second (rad-s-1) and since radian is dimensionless unit of angle so dimension of angular velocity is $\left[\mathrm{T}^{-1}\right]$
Instead of radians angles are often expressed in degrees. So angular velocity can also be expressed in terms of degree per second and degree per minute
we know that
$2 \pi$ radians $=360^{\circ}$
or $\pi$ radians $/ 180^{\circ}=1$ And this relation can be used for expressing angular velocity in degree to that of angular velocity in terms of radian
Angular acceleration is the rate of change of angular velocity w.r.t time. Thus for rigid body rotating about a fixed axis

$$
\begin{equation*}
\alpha=\frac{d \omega}{d t}=\frac{d^{2} \theta}{d t^{2}} \tag{5}
\end{equation*}
$$

Unit of angular acceleration is radian $/ \mathrm{sec}^{2}$
Angular acceleration holds not only for that rotating rigid body but also for the each and every particle of that body
Differentiating equation (4) w.r.t to $t$ we find

$$
\begin{equation*}
a_{i}=\frac{d v_{i}}{d t}=r_{i} \frac{d^{2} \theta}{d t^{2}}=r_{i} \frac{d \omega}{d t} \tag{6}
\end{equation*}
$$

where $\mathrm{a}_{\mathrm{i}}=\mathrm{a}_{\mathrm{it}}=\mathrm{r}_{\mathrm{i}} \alpha$ is the tangential component of linear acceleration of a point at a distance $\mathrm{r}_{\mathrm{i}}$ from the axis Each particle in the rigid body also has the radial linear acceleration component $v^{2} / r$, which can also be expressed in terms of an of angular velocity i.e.

$$
\begin{equation*}
a_{i c}=\frac{v_{i}^{2}}{r_{i}}=r_{i} \omega^{2} \tag{7}
\end{equation*}
$$

and this acceleration $\mathrm{a}_{\mathrm{ic}}$ pointing inwards towards the radial line is also known as centripetal acceleration

## (3) Rotation with constant angular acceleration

We have already studied motion with constant acceleration while studying translational motion
Now we will study the rotational motion with constant angular acceleration
when a rigid body rotates with constant acceleration we have
$\frac{d \omega}{d t}=\alpha=$ cons $\tan t$
Thus we have
Or
$d \omega=\alpha d t$
Integrating on both side
$\int d \omega=\alpha \int d t$
Or
$\omega=\alpha t+c$
If at $\mathrm{t}=0, \omega=\omega_{0}$ then $c=\omega_{0}$ and
$\omega=\omega_{0}+\alpha t$
Again we have angular velocity
$\omega=\frac{d \theta}{d t}$
From equation (8), we have
$d \theta=\omega_{0} d t+\alpha t d t$
Integrating both the sides we have
$\theta=\omega_{0} t+\frac{1}{2} \alpha t^{2}+c_{1}$
If at $\mathbf{t}=0 \theta=\theta_{0}$ then $c_{1}=\theta_{0}$
And above equation becomes
$\theta=\theta_{0}+\omega_{0} t+\frac{1}{2} \alpha t^{2}$
let us write
$\alpha=\omega \frac{\alpha}{\omega}=\omega \frac{d \omega / d t}{d \theta / d t}=\omega \frac{d \omega}{d \theta}$
$\Rightarrow \alpha d \theta=\omega d \omega$
Integrating on both sides
$\int \alpha d \theta=\int \omega d \omega$
$\alpha \theta=\frac{1}{2} \omega^{2}+c_{2}$
At $\mathrm{t}=0 \quad \theta=\theta_{0}$ and $\omega=\omega_{0}$
$\Rightarrow c_{2}=\alpha \theta_{0}-\frac{1}{2} \omega_{0}^{2}$
Putting this value of $c_{2}$ in above equation we have

$$
\begin{equation*}
\omega^{2}=\omega_{0}^{2}+2 \alpha\left(\theta-\theta_{0}\right) \tag{10}
\end{equation*}
$$

Equation (8), (9) and (10) are the equation of motion with constant angular acceleration

## (4) Kinetic energy of Rotation

From equation(4) we know that magnitude of velocity of the ith particle in a rigid body rotating about a fixed axis is
$v_{i}=r_{i}(d \theta / d t)=r_{i} \omega$ where $r_{i}$ is the distance of particle from the axis of rotation and $\omega$ is the velocity of the particle

Kinetic energy of the i particle of mass $\mathrm{m}_{\mathrm{i}}$ is given by
$K_{i}=\frac{1}{2} m_{i} v_{i}^{2}=\frac{1}{2} m_{i} r_{i}^{2} \omega^{2}$
The total kinetic energy of the rigid body as a whole would be equal to the sum of KEs of all particles in the body thus
$K=\sum_{i} K_{i}=\sum_{i} \frac{1}{2} m_{i} r_{i}^{2} \omega^{2}$
Since angular velocity $\omega$ is same for all the particles in the body so
$K=\frac{1}{2}\left[\sum_{i} m_{i} r_{i}^{2}\right] \omega^{2}$
The quantity
$I=\sum_{i} m_{i} r_{i}^{2}$
tell us how the mass of the rotating body is distributed about the axis of rotation is known as rotational inertia or moment of inertia of the rotating body

Moment of inertia of the rigid body can be obtained by imaging the body to be subdivided into large number of particles ,the mass of the each particle is then multiplexed by its squared distance from the axis and then summing over these products for all the particles in the body
Si unit of moment of inertia is $\mathrm{Kgm}^{2}$
So rotational kinetic energy of a body can now be written as
$K E=\frac{1}{2} I \omega^{2}$
Above expression of rotational kinetic energy KE of a rotating rigid body is analogous to the translational kinetic energy where moment of inertia is analogous to mass $m$ (or inertia) and angular velocity is analogous to velocity v
Moment of inertia not only depends on the mass but also on how this mass is distributed about the axis of rotation and it must be specified first before calculating moment of inertia of any body If the body under consideration is of continuous distribution of matter instead of discrete then moment of inertia is evaluated by means of integration rather then that by summation and this point will be discussed in more detail in our next topic

## (5)Calculation of moment of inertia

We already know that the moment of inertia of a system about axis of rotation is given as
$I=\sum_{i} m_{i} r_{i}^{2}$
where $m_{i}$ is the mass of the ith particle and $r_{i}$ is its perpendicular distance from the axis of rotation
For a system consisting of collection of discrete particles ,above equation can be used directly for calculating the moment of inertia

For continuous bodies ,moment of inertia about a given line can be obtained using integration technique For this imagine dividing entire volume of the rigid body into small volume elements dV so that all the points in a particular volume element are approximately at same distance from the axis of rotation and le $r$ be this distance
if dm is the mass of this volume element dV ,then moment of inertia may be given by
$I=\lim _{\Delta m \rightarrow 0} \sum r^{2} \Delta m=\int r^{2} d m$
Since density $\rho$ of the element is defined as mass per unit volume so $\rho=d m / d V$ hence equation (13) may be written as
$I=\int r^{2} \rho d V$
For homogeneous body density $\rho$ is uniform hence $\rho$ can be taken out of the integral sign i.e.
$I=\rho \int r^{2} d V$
above integration can be carried out easily for bodies having regular shapes as can be seen from examples given below
i) Moment of inertia of uniform rod about a perpendicular bisector

Consider a homogeneous and uniform rod of mass $M$ and length $L$ as shown below in the figure


Area of this ring is equal to its circumference multiplied by its width i.e.
Area of the ring $=2 \pi x d x$
Mass of the ring would be
$=\frac{M}{\pi R^{2}} 2 \pi x d x=\frac{2 M x d x}{R^{2}}$
Moment of inertia of this ring about axis OX would be

MI of the ring $=\frac{2 M x d x}{R^{2}} x^{2}=\frac{2 M x^{3} d x}{R^{2}}$
Since whole disc can be supposed to be made up of such like concentric rings of radii ranging from $O$ to $R$,we can find moment of inertia I of the disc by integrating moment of inertia of the ring for the limits $x=0$ and $x=R$
$\therefore M I=\int_{0}^{R} \frac{2 M}{R^{2}} x^{3} d x$
$=\frac{2 M}{R^{2}}\left[\frac{x^{4}}{4}\right]_{0}^{R}=\frac{2 M}{R^{2}} \frac{R^{4}}{4}=\frac{M R^{2}}{2}$
iii) Moment of inertia of a uniform sphere of radius R about the axis through its center

Consider a sphere of mass M and radius R . Let us divide this sphere into thin discs as shown in the figure


Figure 6. Solid uniform sphere of mass M and radius R

If $r$ is the distance of the disc then
$r=\sqrt{\left(R^{2}-x^{2}\right)}$
Volume of the disc would be
$d V=\pi^{2} d x=\pi\left(R^{2}-x^{2}\right) d x$
and its mass would be
$d m=\rho d V$
Moment of inertia of this disc would be
$d I=\frac{\pi \rho}{2}\left(R^{2}-x^{2}\right)^{2} d x$
Moment of inertia of the whole sphere would be
$I=2 \frac{\pi \rho}{2} \int_{0}^{R}\left(R^{2}-x^{2}\right)^{2} d x$
Factor 2 appears because of symmetry considerations as the right hemisphere has same MI as that of left one Integration can be carried out easily by expanding $\left(R^{2}-x^{2}\right)^{2}$.On integrating above equation we find

$$
I=\frac{8 \pi \rho}{15} R^{5}
$$

Now mass of the sphere is
$M=\rho V=\frac{4 \pi \rho R^{3}}{3}$
Hence
$I=\frac{2}{5} M R^{2}$

## (6) Theorems of Moment of Inertia

There are two general theorems which proved themselves to be of great importance on moment of inertia These enable us to determine moment of inertia of a body about an axis if moment of inertia of body about some other axis is known
i) Perpendicular axis theorem

This theorem is applicable only to the plane laminar bodies
This theorem states that, the moment of inertia of a plane laminar about an axis perpendicular to its plane is equal to the sum of the moment of inertia of the lamina about two axis mutually perpendicular to each other in its plane and intersecting each other at the point where perpendicular axis passes through it Consider plane laminar body of arbitrary shape lying in the $x-y$ plane as shown below in the figure


Figure 7. Plane laminar body with z-axis perpendicular to the plane

The moment of inertia about the $z$-axis equals to the sum of the moments of inertia about the $x$-axis and $y$ axis To prove it consider the moment of inertia about $x$-axis
$I_{x}=\sum_{i} m_{i} x_{i}^{2}$
where sum is taken over all the element of the mass $m_{i}$
The moment of inertia about the y axis is
$I_{y}=\sum_{i} m_{i} y_{i}^{2}$
Moment of inertia about $z$ axis is
$I_{z}=\sum_{i} m_{i} r_{i}^{2}$
where $r_{i}$ is perpendicular distance of particle at point $P$ from the $O Z$ axis
For each element
$r_{i}{ }^{2}=x_{i}^{2}+y_{i}^{2}$
$I_{z}=\sum_{i} m_{i} r_{i}^{2}=\sum m_{i}\left(x_{i}^{2}+y_{i}^{2}\right)=\sum_{i} m_{i} x_{i}^{2}+\sum_{i} m_{i} y_{i}^{2}=I_{x}+I_{y}$

## ii) Parallel axis theorem

This theorem relates the moment of inertia about an axis through the center of mass of a body about a second parallel axis
Let $I_{\mathrm{cm}}$ be the moment of inertia about an axis through center of mass of the body and I be that about a parallel axis at a distance $r$ from $C$ as shown below in the figure


Figure 8. Rigid body rotating about an axis parallel to the axis through center of mass at a distance $r$ from it

Then according to parallel axis theorem
$\mathrm{I}=\mathrm{I}_{\mathrm{cm}}+\mathrm{Mr}^{2}$ where M is the total mass of the body
Consider a point $P$ of the body of mass $m_{i}$ at a distance $x_{i}$ from $O$
$\therefore \sum m_{i} x_{i}^{2}=\sum m_{i} C P^{2}+\sum m_{i} r^{2}+2 r \sum m_{i} C Q$
From point $P$ drop a perpendicular PQ on to the OC and join PC.So that
$O P^{2}=C P^{2}+O C^{2}+2 O C . C Q$ ( From geometry) and $m_{i} O P^{2}=m_{i} C P^{2}+m_{i} O C^{2}+2 m_{i} O C . C Q$
$I=I_{c m}+M r^{2}+2 r \sum m_{i} C Q$
Since the body always balances about an axis passing through center of mass, so algebraic sum of the moment of the weight of individual particles about center of mass must be zero. Here
$\sum m_{i} C Q=0$
which is the algebraic sum of such moments about $C$ and therefore eq as $g$ is constant
Thus we have $\mathrm{I}=\mathrm{I}_{\mathrm{cm}}+\mathrm{Mr}^{2}---(17)$

## Torque

Consider two forces $F_{1}$ and $F_{2}$ having equal magnitude and opposite direction acting on a stick placed on a horizontal table as shown below in the figure


Figure 9. Two forces acting on stick are equal in magnitude but opposite in direction tends to rotate the stick

Here note that line of action of forces $F_{1}$ and $F_{2}$ is not same. So they tend to rotate the stick in clockwise direction
This tendency of the force to rotate an object about some axis is called torque
Torque is the rotational counterpart of force. torque tends to rotate an body in the same way as force tends to change the state of motion of the body
Figure below shows a rigid body pivoted at point $O$ so that point $O$ is fixed in space and the body is free to rotate


Figure 10. Force exerted on rigid body pivoted at point O produces torque

Let P be the point of application of force. This force acting at point P makes an angle $\theta$ with the radius vector $\mathbf{r}$ from point O to P
This force F can be resolved into two components
$\mathrm{F}_{\square}=\mathrm{F} \sin \theta$
$\mathrm{F}_{\|}=\mathrm{F} \cos \theta$
as they are perpendicular and parallel to $r$
Parallel component of force does not produce rotational motion of body around point $O$ as it passes through $O$ Effect of perpendicular components producing rotation of rigid body through point O depends on magnitude of the perpendicular force and on its distance r from O
Mathematically ,torque about point O is defined as product of perpendicular component of force and r i.e.
$\mathrm{r}=\mathrm{F}_{\mathrm{q}} \mathrm{r}=\mathrm{F} \sin \theta \mathrm{r}=\mathrm{F}(\mathrm{rin} \theta)=\mathrm{Fd}$
where $d$ is the perpendicular distance from the pivot point ) to the line of action of force $F$ Quantity $d=$ rin $\theta$ is called moment arm or liner arm of force $F$.If $d=0$ the there would be no rotation Torque can either be anticlockwise or clockwise depending on the sense of rotation it tends to produce Unit of torque is Nm
Consider the figure given below where a rigid body pivoted at point $O$ is acted upon by the two force $F_{1}$ and $F_{2}$ $d_{1}$ is the moment arm of force $F_{1}$ and $d_{2}$ is the moment arm of force $F_{2}$


Figure 11. Force $F_{1}$ tends to rotate body in anticlockwise direction and $F_{2}$ in clockwise direction

Force $F_{2}$ has the tendency to rotate rigid body in clockwise direction and $F_{1}$ has the to rotate it in anti clockwise direction
Here we adopt a convention that anticlockwise moments are positive and clockwise moment are negative hence moment $T_{1}$ of force $F_{1}$ about the axis through $O$ is
$\mathrm{T}_{1}=\mathrm{F}_{1} \mathrm{~d}_{1}$
And that of force $F_{2}$ would be
$\mathrm{T}_{2}=-\mathrm{F}_{2} \mathrm{~d}_{2}$
Hence net torque about O is

$$
\begin{aligned}
& \mathrm{T}_{\text {total }}=\mathrm{T}_{1}+\mathrm{T}_{2} \\
& =\mathrm{F}_{1} \mathrm{~d}_{1}-\mathrm{F}_{2} \mathrm{~d}_{2}
\end{aligned}
$$

Rotation of the body can be prevented if
$\mathrm{T}_{\text {total }}=0$
or $\mathrm{T}_{1}=-\mathrm{T}_{2}$
We earlier studied that when a body is in equilibrium under the action of several coplanar forces ,the vector sum of these forces must be zero i.e.
$\Sigma \mathrm{F}_{\mathrm{x}}=0$ and $\Sigma \mathrm{F}_{\mathrm{y}}=0$
we know state our second condition for static equilibrium of rigid bodies that is
" For static equilibrium of rigid body net torque in clockwise direction must be equal to net torque in anticlockwise direction w.r.t some specified axis i.e.
$\Sigma_{\mathrm{T}}=0$

Thus for static equilibrium of an rigid body
i) The resultant external force must be zero
$\Sigma \mathrm{F}=0$
ii) The resultant external torque about any point or axis of rotation must be zero i.e.
$\Sigma_{\mathrm{T}}=0$

## (8) work and power in rotational motion

We know that when we apply force on any object in direction of the displacement of the object ,work is said to be done
Similarly force applied to the rotational body does work on it and this work done can be expressed in terms of
moment of force (torque) and angular displacement $\theta$
Consider the figure given below where a force F acts on the wheel of radius R pivoted at point O .so that it can rotate through point O


Figure 12. Wheel of radius R pivoted at point O
This force F rotates the wheel through an angle $\mathrm{d} \theta$ and $\mathrm{d} \theta$ is small enough so that we can regard force to be constant during corresponding time interval dt
Workdone by this force is
dW=Fds
but $d s=R d \theta$
So
dW=FRd $\theta$
Now FR is the torque $T$ due to force $F$.so we have
$\mathrm{dW}=\mathrm{Td} \theta$----(19)
if the torque is constant while angle changes from $\theta_{1}$ to $\theta_{2}$ then
$\mathrm{W}=\mathrm{T}\left(\theta_{2}-\theta_{1}\right)=\mathrm{T} \Delta \theta---(20)$
Thus workdone by the constant torque equals the product of the torque and angular displacement we know that rate of doing work is the power input of torque so
$\mathrm{P}=\mathrm{dW} / \mathrm{dt}=\mathrm{T}(\mathrm{d} \theta / \mathrm{dt})=\mathrm{T} \omega$
In vector notation

## (9) Torque and angular acceleration

While discussing and defining torque or moment of force, we found that necessary condition for a body not to rotate is that resultant torque about any point should be zero
However this condition is necessary but not sufficient for a rigid body to be static for example in absence of resultant torque a body once set in rotation will continue to rotate with constant angular velocity
Analogous to translation motion when torque acts on a rigid body rotating about a point with constant angular velocity then angular velocity of the body does not remain constant but changes with angular acceleration $\alpha$ which is proportional to the externally applied torque
Consider a force $\mathrm{F}_{\mathrm{i}}$ acting on the ith particle of mass $\mathrm{m}_{\mathrm{i}}$ of the rigid body pivoted about an axis through point O as shown below in the figure


## Figure 13. Force F acting on ith particle of body at point $P$

This force $F_{i}$ as discussed earlier has two components one parallel to the radius vector $r_{i}$ and one perpendicular to the $\mathbf{r}_{\mathbf{i}}$
Component of force parallel to radius vector does not have any effect on the rotation of the body Component of force $F_{i}$ perpendicular does affect the rotation of the body and produces torque about point $O$
through which the body is pivoted which is given by
$\mathrm{T}_{\mathrm{i}}=\mathrm{F}_{\mathrm{i} \square} \mathrm{r}_{\mathrm{i}}---(21)$
if $\mathrm{F}_{\mathrm{i} \square}$ is the resultant force acting on the ith particle ,then from Newton $\square \mathrm{s}$ second law of motion
$F_{i \square}=m_{i} a_{i \square}=m_{i} r_{j} \alpha----(22)$
where $\mathrm{a}_{\mathrm{i}}$ is the tangential acceleration of the body
From equation (21) and (22)

And taking sum over all the particles in the body we have
$\sum \mathrm{T}_{\mathrm{i}}=\sum\left(\mathrm{m}_{\mathrm{i}} \mathrm{r}_{\mathrm{i}}^{2} \alpha\right)=\alpha \sum\left(\mathrm{m}_{\mathrm{i}} \mathrm{r}_{\mathrm{i}}^{2}\right)---(23)$
as angular acceleration is same for all the particles of the body
we know that
$\sum\left(m_{i} r_{i}^{2}\right)=l$
where I is the moment of inertia of the rigid body. Hence in terms of moment of inertia equation 23 becomes
$\sum \mathrm{r}=1 \alpha$
we have denoted resultant torque acting on the body $\sum \mathrm{Tsub}>\mathrm{i}$ as $\sum \mathrm{T}$
Both the torque and angular acceleration are vector quantities so in vector form
$\sum \mathrm{T}=1 \boldsymbol{\alpha}$
Alternatively equation (24) which is rotational analogue of Newton second law of motion ( $\sum \mathrm{F}=\mathrm{ma}$ ) can be written as
$\sum \mathbf{r}=\mid \boldsymbol{\alpha}=\mathrm{I}(\mathrm{d} \boldsymbol{\omega} / \mathrm{dt})=\mathrm{d}(\mathrm{l} \boldsymbol{\omega}) / \mathrm{dt}$
which is similar to the equation
F=d(mv/dt=dp/dt
where $\mathbf{p}$ is the linear momentum
The quantity $\boldsymbol{I} \boldsymbol{\omega}$ is defined as the angular momentum of the system of particles
Angular momentum $=\mathbf{l} \boldsymbol{\omega}$
$L=\mid \omega$
From equation 26 we see that resultant torque acting on a system of particles equal to the rate of change of the angular momentum
$\sum \mathrm{r}=\mathrm{d} \mathrm{L} / \mathrm{dt}$
(10) Angular momentum and torque as vector product
(A) Angular momentum

In any inertial frame of refrance the moment of linear momentum of a particle is known as angular momentum or, angular momentum of a particle is defined as the moment of its linear momentul.

In rotational motion angular momentum has the same significance as linear momentum have in the linear motion of a particle.
Value of angular momentum of angular momentum is equal to the product of linear momentum and $\mathbf{p}(=m \mathbf{v})$ and the position vector $r$ of the particle from origin of axis of rotation.


Figure 14

Angular mmomentum vector is usually represented by $\mathbf{L}$.
If the linear momentum of any particle is $\mathbf{p}=\mathbf{m v}$ and its position vector from any constant point be $\mathbf{r}$ then abgular momentum of the particle is given by
$\mathbf{L}=\mathbf{r} \times \mathbf{p}=\mathbf{m}(\mathbf{r} \times \mathbf{v})$
Angular momentum is a vector quantity and its direction is perpandicular to the direction of $\mathbf{r}$ and $\mathbf{p}$ and could be found out by right hand screw rule.
From equation 1 scalar value or magnitude of angular momentum is given as
|니=rpsin $\theta$
where V is the angle between $\mathbf{r}$ and $\mathbf{p}$.
For a particle moving in a circular path
$\mathbf{v}=\boldsymbol{\omega} \times \mathbf{r}$;
where $\boldsymbol{\omega}$ is the angular velocity.


Figure 15

Therefore
$\mathbf{L}=\mathrm{m}[\mathbf{r} \times(\boldsymbol{\omega} \times \mathbf{r})]=\mathrm{m}\{\boldsymbol{\omega}(\mathbf{r} . \mathbf{r})-\mathbf{r}(\mathbf{r} . \boldsymbol{\omega})\}=m r^{2} \boldsymbol{\omega}=\boldsymbol{\omega}$;
$(\mathbf{r} . \boldsymbol{\omega})=0$ because in circular motion $\mathbf{r}$ and $\boldsymbol{\omega}$ are perpandicular to each other. Here $I$ is the moment of inertia of the particle about the given axis also the direction of $L$ and $\boldsymbol{\omega}$ is same and this is a axial vector. writing equation 1 in the component form we get
$L=\boldsymbol{r} \times \boldsymbol{p}=\left|\begin{array}{ccc}i & j & k \\ \mathrm{x} & \mathrm{y} & \mathrm{z} \\ \mathrm{p}_{\mathrm{x}} & \mathrm{p}_{\mathrm{y}} & \mathrm{p}_{z}\end{array}\right|=i\left(\mathrm{yp}_{z}-\mathrm{zp}_{\mathrm{y}}\right)+j\left(\mathrm{zp}_{\mathrm{x}}-\mathrm{xp}_{z}\right)+\boldsymbol{k}\left(\mathrm{xp}_{y}-\mathrm{yp}_{\mathrm{x}}\right)$
Writing angular momentum in component form we get
$\boldsymbol{L}=\boldsymbol{i} \mathrm{L}_{\mathrm{x}}+j \mathrm{~L}_{\mathrm{y}}+\boldsymbol{k} \mathrm{L}_{z}$
writing equation 5 again we get
$i \mathrm{~L}_{\mathrm{x}}+j \mathrm{~L}_{\mathrm{y}}+\boldsymbol{k} \mathrm{L}_{z}=i\left(\mathrm{yp}_{z}-\mathrm{zp} \mathrm{p}_{y}\right)+j\left(\mathrm{zp}_{\mathrm{x}}-\mathrm{xp}_{z}\right)+\boldsymbol{k}\left(\mathrm{xp}_{\mathrm{y}}-\mathrm{yp}_{\mathrm{x}}\right)$
Comparing unit vectors on both the sides we get
$\mathrm{L}_{\mathrm{x}}=\left(\mathrm{yp}_{\mathrm{z}}-\mathrm{zp}_{\mathrm{y}}\right)$
$\mathrm{L}_{\mathrm{y}}=\left(\mathrm{zp}_{\mathrm{x}}-\mathrm{xp} \mathrm{p}_{\mathrm{z}}\right)$
$L_{z}=\left(\mathrm{xp}_{\mathrm{y}}-\mathrm{yp}_{\mathrm{x}}\right)$
Unit of angular momentum in CGS is gm. $\mathrm{cm}^{2} / \mathrm{sec}$ and in MKS system it is $\mathrm{Kgm} . \mathrm{m}^{2} / \mathrm{sec}$ or Joule/sec.

## (B) Torque

The turning effect of the force about the axis of rotation is called the moment of force or torque..
In rotational motion torque has same importance as that of force in the linear motion.
Torque due to a force $\mathbf{F}$ is measured as a vector product of force $\mathbf{F}$ and position vector $\mathbf{r}$ of line of action of force from the axis of rotation.
We already know that orque is denoted by letter $\mathbf{t}$.


Figure 16
If $\mathbf{F}$ is the force acting on the particle and $\mathbf{r}$ is the position vector of particle with respect to constant point then the torque acting on the particle is given by
$\mathbf{r}=\mathbf{r} \times \mathrm{F}$
FRom equation 8 magnitude or resultant of torque is given by
$|\mathrm{T}|=\mathrm{rfsin} \theta$
where $\theta$ is the angle between $\mathbf{r}$ and $\mathbf{F}$.
From equation 9 if $\theta=90^{\circ}$ this menas $r$ is perpandicular to $F$ then,
FRom equation 8 magnitude or resultant of torque is given by
$|\boldsymbol{T}|=r F$
and if $\theta=0^{0}$ this menas $\boldsymbol{r}$ is parallel to $\mathbf{F}$ then,
$|\boldsymbol{T}|=0$
Unit of torque is Dyne-cm or Newton-m

## (C) Relation between angular momentum and torque

Differentiating equation 1 w.r.t. t we get

$$
\frac{\mathrm{d} \boldsymbol{L}}{\mathrm{dt}}=\frac{\mathrm{d}(\boldsymbol{r} \times \boldsymbol{p})}{\mathrm{dt}}=\left(\frac{\mathrm{d} \boldsymbol{r}}{\mathrm{dt}} \times \boldsymbol{p}\right)+\left(\boldsymbol{r} \times \frac{\mathrm{d} \boldsymbol{p}}{\mathrm{dt}}\right)
$$

but, $\frac{\mathrm{d} \boldsymbol{r}}{\mathrm{dt}}=\boldsymbol{v}$ and, $p=\mathrm{m} v$
hence, $\left(\frac{\mathrm{d} \boldsymbol{r}}{\mathrm{dt}} \times \boldsymbol{p}\right)=\boldsymbol{v} \times \mathrm{m} \boldsymbol{v}=\mathrm{m}(\boldsymbol{v} \times \boldsymbol{v})=0$
Therefore,
$\frac{\mathrm{d} L}{\mathrm{dt}}=\left(r \times \frac{\mathrm{d} p}{\mathrm{dt}}\right)$
But from Newton's second law of motion we have

## (11)Angular momentum and torque of the system of particles

## (A) Angular momentum of system of particles

Consider a system of particles made up of number of particles, moving independently to each other. Let $L_{1}$, $\mathrm{L}_{2}, \mathrm{~L}_{3}$ $\qquad$ etc. be the angular momentum of different particles of the system w.r.t., to a given point. The angulat momentum of the particle system w.r.t. that given point is equal to the vector sum of angular momentum of all the particles of the system.
If L is the angular momentum of the system of particles or the body as a whole then,
$\boldsymbol{L}=\boldsymbol{L}_{1}+\boldsymbol{L}_{2}+\boldsymbol{L}_{3}+\ldots \ldots \ldots+\boldsymbol{L}_{n}=\left(r_{1} \times \mathrm{m}_{1} \boldsymbol{v}_{1}\right)+\left(r_{2} \times \mathrm{m}_{2} \boldsymbol{v}_{2}\right)$
or,
$\boldsymbol{L}=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\boldsymbol{r}_{\mathrm{i}} \times \mathrm{m}_{\mathrm{i}} v_{i}\right)=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\boldsymbol{r}_{\mathrm{i}} \times \boldsymbol{p}_{\mathrm{i}}\right)$
(B) Torque acting on system of particles

Torque acting on the system of particles from equation above angular momentem of the system of particles is given as
$\mathbf{L}=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathbf{r}_{\mathrm{i}} \times \mathbf{p}_{\mathrm{i}}\right)$
Differentiating above equation with respect to time $t$ we get
$\frac{\mathrm{d} \mathbf{L}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}}\left[\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathbf{r}_{\mathbf{i}} \times \mathbf{p}_{\mathrm{i}}\right)\right]=\sum_{i=1}^{n}\left[\frac{d \mathbf{r}_{\mathrm{i}}}{d t} X \mathbf{p}_{\mathrm{i}}+\mathbf{r}_{\mathrm{i}} X \frac{d \mathbf{p}_{\mathrm{i}}}{d t}\right]$
But $\frac{d \mathbf{r}_{\mathbf{i}}}{d t}=\mathbf{v}_{\mathbf{i}}$ and $\mathbf{p}_{\mathbf{i}}=m \mathbf{v}_{\mathbf{i}}$
Hence $\frac{\mathrm{d} \mathbf{r}_{\mathbf{i}}}{\mathrm{dt}} \times \mathbf{p}_{\mathbf{i}}=\mathbf{v}_{\mathbf{i}} \times \mathrm{m} \mathbf{v}_{\mathbf{i}}=\mathrm{m}\left(\mathbf{v}_{\mathbf{i}} \times \mathbf{v}_{\mathbf{i}}\right)=0$
And $\frac{d \mathbf{p}_{\mathrm{i}}}{d t}=\mathbf{F}_{\mathrm{i}}$
$\therefore \frac{\mathrm{d} \mathbf{L}}{\mathrm{dt}}=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathbf{r}_{\mathbf{i}} \times \mathbf{F}_{\mathrm{i}}\right)$
If the torque acting on the system of particles is $\tau$ then
$\tau=\frac{\mathrm{d} \mathbf{L}}{\mathrm{dt}}=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathbf{r}_{\mathrm{i}} \times \mathbf{F}_{\mathrm{i}}\right)$
When particles of the system are in motion then their motion is due to external and interaction due to internal forces so force acting on any particle of the system is given by
$\mathbf{F}_{\mathbf{i}}=\mathbf{F}_{\mathrm{iext}}+\sum_{j=1}^{n} \mathbf{F}_{\mathrm{ij}}$
Here $F_{\text {iext }}$ is the external force acting on the ith particle and $\sum F_{i j}$ is the sum of the force acting on the particle due to internal interaction of different particles.Putting the value of $F_{i}$ in the equation we get
$\tau=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{r}_{\mathrm{i}} \times\left(\mathbf{F}_{\text {iext }}+\sum_{\mathrm{j}=1}^{\mathrm{n}} \mathbf{F}_{\mathrm{ij}}\right)=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{r}_{\mathbf{i}} \times \mathbf{F}_{\mathrm{iext}}+\sum_{\mathrm{i}=1}^{\mathrm{n}} \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathbf{r}_{\mathbf{i}} \times \mathbf{F}_{\mathrm{ij}}$
RHS of the equation 2 shows that summantion of the moment of interacting force (internal). Here internal interaction forces balance each other so torque due to internal forces adds to zero hence

$$
\begin{align*}
& \sum_{i=1}^{\mathrm{n}} \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathbf{r}_{\mathrm{i}} \times \mathbf{F}_{\mathrm{ij}}=0 \\
& \therefore \tau=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{r}_{\mathrm{i}} \times \mathbf{F}_{\text {iext }} \\
& \tau=\sum_{\mathrm{i}=1}^{\mathrm{n}} \tau_{\mathrm{iext}}  \tag{6}\\
& \tau=\frac{\mathrm{d} \mathbf{L}}{\mathrm{dt}}=\tau_{\text {ext }} \tag{7}
\end{align*}
$$

Above equation proves that total torque acting on a system of particles is equal to the vector sum of the torque acting on the different particles due to external force on the particle and its value is also equal to the rate of change of angular momentum

## (12) Angular momentum of the system of particles with respect to the center of mass of the system

let a system of particles is made up of $n$ number of particles. Let $\mathbf{r}_{\mathbf{i}}$ be the position vector of the ith particle $P$ with respect to a poiint $O$ and $\mathbf{v}_{\mathbf{i}}$ be its velocity . let $\mathbf{R}_{\mathbf{c m}}$ be the position vector of center of mass $C$ of the system with respect to the origin
Let $\mathbf{r}_{\mathbf{i}}^{\prime}$ and $\mathbf{v}_{\mathbf{i}}$ ' be the position vector and velocity vector of the ith particle with respect to center of mass of the system.

Angular momentum of the system of particles with respect to origin is given by

$$
\begin{equation*}
\mathbf{L}_{0}=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathbf{r}_{\mathrm{i}} \times \mathrm{m}_{\mathrm{i}} \mathbf{v}_{\mathrm{i}}\right)=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathbf{r}_{\mathrm{i}} \times \mathbf{p}_{\mathrm{i}}\right) \tag{1}
\end{equation*}
$$



Figure 17

Angular momentum of the system of particles with respect to center of mass of the system is given by
$\mathbf{L}_{\mathrm{cm}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}_{\mathrm{i}}\left(\mathbf{r}_{\mathbf{i}}^{\prime} \times \mathbf{v}_{\mathbf{i}}^{\prime}\right)$
From figure we have

$$
\begin{equation*}
\mathbf{r}_{\mathrm{i}}^{\prime}=\mathbf{r}_{\mathrm{i}}-\mathbf{R}_{\mathrm{cm}} \tag{3}
\end{equation*}
$$

Differentiating with respect to time

$$
\begin{align*}
& \frac{d \mathbf{r}_{\mathrm{i}}^{\prime}}{d t}=\frac{d \mathbf{r}_{\mathrm{i}}}{d t}-\frac{d \mathbf{R}_{\mathrm{cm}}}{d t} \\
& \mathbf{v}_{\mathbf{i}}^{\prime}=\mathbf{v}_{\mathbf{i}}-\mathbf{V}_{\mathrm{cm}} \tag{4}
\end{align*}
$$

Putting these values of equation 3 and 4 in equation 2

$$
\begin{aligned}
\mathbf{L}_{\mathrm{cm}} & =\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}\left[\left(\mathbf{r}_{\mathrm{i}}-\mathbf{R}_{\mathrm{cm}}\right) \times\left(\mathbf{v}_{\mathrm{i}}-\mathbf{V}_{\mathrm{cm}}\right)\right] \\
\mathbf{L}_{\mathrm{cm}} & =\sum_{\mathrm{i}=1}^{\mathrm{n}}\left[\mathrm{~m}_{\mathrm{i}}\left(\mathbf{r}_{\mathrm{i}} \times \mathbf{v}_{\mathrm{i}}\right)-\mathrm{m}_{\mathrm{i}}\left(\mathbf{r}_{\mathrm{i}} \times \mathbf{V}_{\mathrm{cm}}\right)-\mathrm{m}_{\mathrm{i}}\left(\mathbf{R}_{\mathrm{cm}} \times \mathbf{v}_{\mathrm{i}}\right)+\mathrm{m}_{\mathrm{i}}\left(\mathbf{R}_{\mathrm{cm}} \times \mathbf{V}_{\mathrm{cm}}\right)\right] \\
& =\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}\left(\mathbf{r}_{\mathrm{i}} \times \mathbf{v}_{\mathrm{i}}\right)-\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}\left(\mathbf{r}_{\mathrm{i}} \times \mathbf{V}_{\mathrm{cm}}\right)-\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}\left(\mathbf{R}_{\mathrm{cm}} \times \mathbf{v}_{\mathrm{i}}\right)+\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}}\left(\mathbf{R}_{\mathrm{cm}} \times \mathbf{V}_{\mathrm{cm}}\right)
\end{aligned}
$$

Since for system of particles value of $\mathbf{R}_{\mathrm{cm}}$ and $\mathbf{V}_{\mathrm{cm}}$ remains constant hence

$$
\begin{equation*}
\therefore \mathbf{L}_{\mathrm{cm}}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathbf{r}_{\mathrm{i}} \times \mathrm{m}_{\mathrm{i}} \mathbf{v}_{\mathrm{i}}-\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \mathbf{r}_{\mathrm{i}}\right) \times \mathbf{V}_{\mathrm{cm}}-\mathbf{R}_{\mathrm{cm}} \times\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \mathbf{v}_{\mathrm{i}}\right)+\left(\mathbf{R}_{\mathrm{cm}} \times \mathbf{V}_{\mathrm{cm}}\right) \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{~m}_{\mathrm{i}} \tag{5}
\end{equation*}
$$

But
$\sum_{i=1}^{n} m_{i} \mathbf{r}_{\mathrm{i}}=m \mathbf{R}_{\mathrm{cm}}$
$\sum_{i=1}^{n} m_{i} \mathbf{v}_{\mathrm{i}}=m \mathbf{V}_{\mathrm{cm}}$
And
$\sum_{i=1}^{n} m_{i}=m$
From the above equation, the angular momentum equation becomes

$$
\begin{align*}
& \mathbf{L}_{\mathrm{cm}}=\mathbf{L}_{0}-\mathrm{m}\left(\mathbf{R}_{\mathrm{cm}} \times \mathbf{V}_{\mathrm{cm}}\right)-\mathbf{R}_{\mathrm{cm}} \times \mathrm{m} \mathbf{V}_{\mathrm{cm}}+\left(\mathbf{R}_{\mathrm{cm}} \times \mathbf{V}_{\mathrm{cm}}\right) \mathrm{m} \\
& \mathbf{L}_{\mathrm{cm}}=\mathbf{L}_{0}-\mathbf{R}_{\mathrm{cm}} \times \mathrm{m} \mathbf{V}_{\mathrm{cm}} \\
& \mathbf{L}_{\mathrm{cm}}=\mathbf{L}_{0}-\mathbf{R}_{\mathrm{cm}} \times \mathbf{p}_{\mathrm{cm}} \tag{7}
\end{align*}
$$

Where $\mathbf{p}_{\mathrm{cm}}$ is the linear momentum of center pf mass in labotary frame of refrence
Or

$$
\begin{equation*}
\mathbf{L}_{\mathrm{cm}}=\mathbf{L}_{0}-\mathbf{L}_{\mathrm{cm} 0} \tag{8}
\end{equation*}
$$

Here $\mathbf{L}_{\mathrm{cm0}}$ is the angular momentum of center of mass w.r.t.O

$$
\begin{equation*}
\mathbf{L}_{0}=\mathbf{L}_{\mathrm{cm}}+\mathbf{L}_{\mathrm{cm} 0} \tag{9}
\end{equation*}
$$

Hence the angular momentum of the system of the particles with respect to point $O$ is equal to the sum of the angular momentum of the center of mass of the particles about $O$ and angular monentum of the system about center of mass

## (13) Law of conservation of angular momentum

Torque acting on any particle is given by
$\tau=\frac{\mathrm{d} L}{\mathrm{dt}}$
If external torque acting on any particle os zero then,
i.e., if $\tau=0$
then, $\frac{\mathrm{d} \mathbf{L}}{\mathrm{dt}}=0$
or, $\mathbf{L}=$ constant
Hence in absence of external torque the angular momentum of the particle remains constant or conserved. Total torque acting on any system is given by
$\boldsymbol{\tau}=\tau_{\text {ext }}=\frac{\mathrm{d} \boldsymbol{L}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \boldsymbol{L}_{i}=\sum_{\mathrm{i}=1}^{\mathrm{n}} \boldsymbol{r}_{i} \times \boldsymbol{F}_{\text {iext }}$
If total external force acting on any particle system is zero or,

$$
\tau_{\mathrm{ext}}=0
$$

then, $\frac{\mathrm{d} \boldsymbol{L}}{\mathrm{dt}}=\frac{\mathrm{d}}{\mathrm{dt}}\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \boldsymbol{L}_{i}\right)=0$
$L=L_{1}+L_{2}+L_{3}+\ldots \ldots . . . . .=$ constant
If total external torque acting on any body is zero, then total angular momentum of the body remains constant or conserved.

## (14) Radius of gyration

Whatever may be the shape of the body it is always possible to find a distance from the axis of rotation at which whole mass of the body can be assumed to be concentrated and even then its moment of inertia about that axis remains unchanged.
If whole mass of the body is supposed to be concentrated at a distance k from the axis of rotation then $\mathrm{I}=\mathrm{Mk}^{2}=\Sigma \mathrm{mr}^{2}$
or ,
$\mathrm{k}=\sqrt{ }(1 / \mathrm{M})=\sqrt{ }\left(\sum m r^{2} / \mathrm{M}\right)$
This quantity k is called radius of gyration of the body about the axis of rotation.
Thus, the radius of gyration of a body, rotating about a given axis of rotation is the radial distance from the axis and when the square of radius of gyration $(k)$ is multiplied by the total mass of the body it gives the moment of inertia of the body about that axis.

## (15) Kinetic Energy of rolling bodies (rotation and translation combined)

Let us now calculate the kinetic energy of a rolling body. For this consider a body with circular symmetry for example cylinder, wheel, disc , sphere etc.
When such a body rolls on a plane surface , the motion of such a body is a combination of translational motion and rotational motion as shown below in the figure.


Figure 18
At any instant the axis normal to the digram through the point of contact $P$ is the axis of rotation. If the speed of the centre of mass relative to an observer fixed on the surface is $\mathrm{V}_{\mathrm{cm}}$ then the instantaneous angular speed about an axis through P would be
$\omega=\mathrm{V}_{\mathrm{cm}} / \mathrm{R}$
where R is the radius of the body.
To explain this consider the figure given below


Figure 19. All points on the rolling object rotates about point of contact $P$

At any instant different particles of the body have different linear speeds. The point $P$ is at rest $\mathrm{V}_{\mathrm{cm}}=0$ instantaneously, the centre of mass has speed $V_{c m}=R \omega$ and the highest point on the circumfrance $\rho^{\prime}$ has speed $V_{c m}=2 R \omega$ relative to point $P$.
Now again consider the first figure the top of the cylinder has linear speed $\mathrm{V}_{\mathrm{cm}}+\mathrm{R} \mathrm{\omega}=\mathrm{V}_{\mathrm{cm}}+\mathrm{V}_{\mathrm{cm}}=2 \mathrm{~V}_{\mathrm{cm}}$, which is greater than the linear speed of any other point on the cylinder. We thus note that the center of mass moves with linear speed $\mathrm{V}_{\mathrm{cm}}$ while the contact point between the surface and rolling objectr has a linear speed of zero. Therefore at that instant all particles of the rigid body are moving with the same angular speed $\omega$ about the
axis through P and the motion of the body is equivalent to pure rotational motion.
Thus total kinetic energy is
$K=1 / 2\left(I_{p} \omega^{2}\right)$
where $I_{P}$ is the moment of inertia of the rigid body about point $P$.
From parallel axis theorem
$\mathrm{I}_{\mathrm{P}}=\mathrm{I}_{\mathrm{Cm}}+\mathrm{MR}^{2}$
where $\mathrm{I}_{\mathrm{cm}}$ is the moment of inertia of the body of mass M about parallel axis through point O .
Therefore
$\mathrm{K}=1 / 2\left(\mathrm{I}_{\mathrm{cm}} \omega^{2}\right)+1 / 2\left(\mathrm{MR}^{2} \omega^{2}\right)=1 / 2\left(\mathrm{I}_{\mathrm{cm}} \omega^{2}\right)+1 / 2\left(\mathrm{M}\left(\mathrm{V}_{\mathrm{cm}}\right)^{2}\right)$
here the first term represents the rotational kinetic energy of the cylinder about its center of mass, and the second term represents the kinetic energy the cylinder would have if it were just translating through space without rotating. Thus, we can say that the total kinetic energy of a rolling object is the sum of the rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass.
If $k$ is the radius of gyration of the body about a parallel axis through O then $\mathrm{I}=\mathrm{Mk}^{2}$ and total kinetic energy would then be,
$\mathrm{K}=\frac{1}{2} \mathrm{Mk}^{2} \frac{\mathrm{~V}_{\mathrm{cm}}^{2}}{\mathrm{R}^{2}}+\frac{1}{2} \mathrm{MV}_{\mathrm{cm}}^{2}$
$\mathrm{K}=\frac{1}{2} \mathrm{MV}_{\mathrm{cm}}^{2}\left(\frac{\mathrm{k}^{2}}{\mathrm{R}^{2}}+1\right)$

## Rotation

# Question 1. A mass is whirled in a circular path with constant angular velocity and its angular momentum is L.If 

 the string is now halve keeping the angular velocity same then angular momentum isa. L
b. L/4
C. L/2
d. 2L

## Solution 1

Angular momentum for this is defined as
$=m r^{2} \omega$

## First case

$\mathrm{L}=\mathrm{mr}^{2} \omega$

## Second case

$L_{f}=m(r / 2)^{2} \omega$

## So $\mathrm{L}_{\mathrm{f}}=\mathrm{L} / 4$

Question 2.A mass is moving with constant velocity along a line parallel to xaxis away from origin.its angular momentum with respect to origin is
a. is zero
b. remains constant
c. goes on increasing
d. goes on decreasing

## Solution 2

$\mathrm{L}=(\mathrm{mv}) \mathrm{Xr}$
or
L=mvrsin $\theta$
Now rsin $\theta=$ perpendicular distance from x axis which is constant
So Angular momentum is constant

Question 3.A cylinder rolls up the incline plane reaches some height and then roll down without slipping through out this section. The direction of the frictional force acting on the cylinder are
a. Up the incline while ascending and down the incline while descending
b.Up the incline while ascending and desending
c. down the incline while ascending and up the incline while descending
d.down the incline while ascending and desending

## Solution 3:

Imagine the cylinder to be moving on a frictionless surface. In both the cases the acceleration of the CM of the cylinder is gsin $\theta$. This is also the acceleration of the point of contact
of the cylinder and the inclined plane..Also no torque (about the center of the cylinder) is acting on the cylinder since we assumed the surface to be a frictionless and the forces
acting on the cylinder is mg and N which passes through the center of the cylinder. Therefore the net movement of the point of contact in both the cases is in downward direction
Therefore frictional force will act in upward direction in both the cases

Question 4.A uniform sold sphere rolls on the horizontal surface at $20 \mathrm{~m} / \mathrm{s}$.it then rolls up the incline of 30 .If friction losses are negligible what will be the value of $h$ where sphere stops on the incline
a. 28.6 m
b 30 m
c. 28 m
d. none of these

## Solution 4

Let h be the height

The rotational and translational KE of the ball at the bottom will be changed to Gravitational energy when the sphere stops .
We therefore writes
$(1 / 2) \mathrm{Mv}^{2}+(1 / 2) \mid \omega^{2}=\mathrm{Mgh}$

For a solid sphere $\mathrm{I}=(2 / 5) \mathrm{Mr}^{2}$ and also $\omega=\mathrm{v} / \mathrm{r}$

Question 5. A cylinder of Mass $M$ and radius $R$ rolls down a incline plane of inclination $\theta$.Find the linear accleration of the cylinder
a. $(2 / 3) g \sin \theta$
b. $(2 / 3) g \cos \theta$
c $g \sin \theta$
d none of these

## Solution 5

Net force on the cylinder
$F_{\text {net }}=m g \sin \theta-f$
or $m a=m g \sin \theta-f$
Where f is the frictional force
Now $\mathrm{t}=\mathrm{fXR}=1 \alpha$
Now in case of pure rolling we know that
$a=\alpha R=>\alpha=a / R$

So $f=l a / R^{2}$

From 1 and 2
$a=m g \sin \theta /\left[m+\left(I / R^{2}\right)\right]$

Now $I=m R^{2} / 2$

So $a=(2 / 3) g \sin \theta$

Question 6 An ice skater spins with arms outstretch at $1.9 \mathrm{rev} / \mathrm{s}$.Her moment of inertia at this time is 1.33 $\mathrm{kgm}^{2}$. She pulls her arms to increase her rate of spin. Her moment of inertia after she pulls her arm is $.48 \mathrm{kgm}^{2}$. What is her new rate of spinning
a. $5.26 \mathrm{rev} / \mathrm{s}$
b. $5.2 \mathrm{rev} / \mathrm{s}$
d. none of thes

## Solution 6

Law of conservation of angular momentum
$l_{1} \omega_{1}=I_{2} \omega_{2}$
or
$1.33(1.9)=.48 \omega_{2}$
or
$\omega_{2}=5.26 \mathrm{rev} / \mathrm{s}$

Question 7. Moment of inertia of a uniform rod of lenght $L$ and mass $M$ about an axis passing through $L / 4$ from one end and perpendicular to its lenght
a. $7 \mathrm{ML}^{2} / 36$
b. $7 \mathrm{ML}^{2} / 48$
c. $11 \mathrm{ML}^{2} / 48$
d. $\mathrm{ML}^{2} / 12$

## Solution 7

Using parallel axis theorem
$I=I_{c m}+M x^{2}$ where $x$ is the distance of the axis of the rotation from the CM of the rod
So $\mathrm{x}=\mathrm{L} / 2-\mathrm{L} / 4=\mathrm{L} / 4$ Also $\mathrm{I}_{\mathrm{cm}}=\mathrm{ML}^{2} / 12$

So $I=\mathrm{ML}^{2} / 12+\mathrm{ML}^{2} / 16=7 \mathrm{ML}^{2} / 48$

Question 8. A wheel starts from rest and spins with a constant angular acceleration. As time goes on the acceleration vector for a point on the rim:
a. increases in magnitude but retains the same angle with the tangent to the rim
b.increases in magnitude and becomes more nearly radial
c. increases in magnitude and becomes more nearly tangent to the rim
d. decreases in magnitude and becomes more nearly radial

## Solution 8

Tangential acceleration=radius* angular acceleration
Since angular acceleration is constant ...Tangential acceleration is constant

Radial acceleration=r* (angular velocity) ${ }^{2}$
Since angular velocity increase with time...Radial acceleration increase with time

So resulttant acceleration increase with time and becomes more radial as time passes

## Question 9.

Two wheels are identical but wheel $B$ is spinning with twice the angular speed of wheel $A$. The ratio of the magnitude of the \&radical acceleration of a point on the rim of $B$ to the magnitude of the radial acceleration of a point on the rim of $A$ is:
a. 4
b. 2
c 1/2
d $1 / 4$

## Solution 9.

Radial acceleration $=r^{*}(\omega)^{2}$

For wheel A
Radial acceleration of $A=r^{*}(\omega)^{2}$

For wheel B
Radial acceleration of $B=r^{*}(2 \omega)^{2}=4 r^{*}(\omega)^{2}$

So Radial acceleration of $B /$ Radial acceleration of $A=4: 1$

Question 10. For a wheel spinning with constant angular acceleration on an axis through its center, the ratio of the speed of a point on the rim to the speed of a point halfway between the center and the rim is:

## Solution 10

At rim
$v=r \omega$

At point between the center and rim
$v=(r / 2) \omega$

Ratio =2

Question 11. A wheel initially has an angular velocity of $18 \mathrm{rad} / \mathrm{s}$. It has a constant angular acceleration of 2.0 $\mathrm{rad} / \mathrm{s} 2$ and is
slowing at first. What time elapses before its angular velocity is $18 \mathrm{rad} / \mathrm{s}$ in the direction opposite to its initial angular velocity?
a 3 sec
b 6 sec
c 18 sec
d none of these

## Solution 11

$\omega_{0}=18$
$\omega=-18$
anugular acceleration $(\alpha)=-2$

Now
$\omega=\omega_{0}+\alpha t$
or $t=18$

Question 12. One solid sphere $X$ and another hollow sphere $Y$ are of same mass and same outer radii. Their moment of inertia about their diameters are respectively $\mathrm{I}_{\mathrm{x}}$ and $\mathrm{I}_{\mathrm{y}}$ such that
(A) $I_{x}=I_{y}$
(B) $I_{x}>I_{y}$

## (D) $I_{x} / I_{y}=D_{x} / D_{y}$

 Where $D_{x}$ and $D_{y}$ are their densities.
## MOTION OF SYSTEM OF PARTICLES AND RIGID BODY

## CONCEPTS.

.Centre of mass of a body is a point where the entire mass of the body can be supposed to be concentrated

For a system of $n$-particles, the centre of mass is given by

$$
\vec{r}=\frac{m_{1} \vec{r}_{1}+m_{2} \overrightarrow{r_{2}}+m_{3} \overrightarrow{r_{3}}+\ldots \ldots . .+m_{n} \overrightarrow{r_{n}}}{m_{1}+m_{2}+m_{3}+\ldots \ldots+m_{n}}=\frac{\sum_{i=1}^{i=n} m_{i} \vec{r}_{1}}{M}
$$

.Torque $\tau$ The turning effect of a force with respect to some axis, is called $\square$ moment of force or torque due to the force. Torque is measured as the $\square$ product of the magnitude of the force and the perpendicular distance of $\square$ the line of action of the force from the axis of rotation.

$$
\vec{\tau}=\vec{r} \times \vec{F}
$$

.Angular momentum $(\vec{L})$. It is the rotational analogue of linear momentum and is measured as the product of linear momentum and the perpendicular distance of its line of axis of rotation.

Mathematically: If $\vec{P}$ is linear momentum of the particle and $\vec{r}$ its position vector, then angular momentum of the particle, $\vec{L}=\vec{r} \times \vec{P}$
(a)In Cartesian coordinates: $L_{Z}=x p_{y}-y p_{x}$
(b)In polar coordinates : $L=r p \sin \emptyset$,

Where $\emptyset$ is angle between the linear momentum vector $\vec{P}$ and the position of vector $\vec{r}$.
S.I unit of angular momentum is $\mathrm{kg} m^{2} s^{-1}$.

Geometrically, angular momentum of a particle is equal to twice the product of mass of the particle and areal velocity of its radius vector about the given axis.
.Relation between torque and angular momentum:
$\begin{array}{ll}\text { (i) } \vec{\tau}=\frac{d \vec{L}}{d t} & \text { (ii) If the system consists of n-particles, then } \vec{\tau}=\frac{d \vec{L}_{1}}{d t}+\frac{d \vec{L}_{2}}{d t}+\frac{d \vec{L}_{3}}{d t}+ \\ \cdots+\frac{d \vec{L}_{n}}{d t} .\end{array}$
.Law of conservation of angular momentum. If no external torque acts on a system, then the total angular momentum of the system always remain conserved.

Mathematically: $\vec{L}_{1}+\vec{L}_{2}+\vec{L}_{3}+\cdots+\vec{L}_{n}=\vec{L}_{\text {total }}=a$ constant
.Moment of inertia(I).the moment of inertia of a rigid body about a given axis of rotation is the sum of the products of masses of the various particles and squares of their respective perpendicular distances from the axis of rotation.

Mathematically: $\mathrm{I}=m_{1} r_{1}^{2}+m_{2} r_{2}^{2}+m_{3} r_{3}^{2}+\ldots+m_{n} r_{n}^{2}=\sum_{i=1}^{i=n} m_{i} r_{i}^{2}$
SI unit of moment of inertia is $\mathrm{kg} \mathrm{m}^{2}$.
MI corresponding to mass of the body. However, it depends on shape \& size of the body and also on position and configuration of the axis of rotation.

Radius of gyration (K).it is defined as the distance of a point from the axis of rotation at which, if whole mass of the body were concentrated, the moment of inertia of the body would be same as with the actual distribution of mass of the body.

Mathematically : $\mathrm{K}=\frac{r_{1}^{2}+r_{2}^{2}+r_{3}^{2}+\cdots+r_{n}^{2}}{n}=$ rms distance of particles from the axis of rotation.

SI unit of gyration is m . Note that the moment of inertia of a body about a given axis is equal to the product of mass of the body and squares of its radius of gyration about that axis i.e. $\mathrm{I}=\mathrm{M} k^{2}$.
.Theorem of perpendicular axes. It states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moment of
inertia of the lamina about any two mutually perpendicular axes in its plane and intersecting each other at the point, where the perpendicular axis passes through the lamina.

Mathematically: $I_{z}=I_{x}+I_{y^{\prime}}$
Where $x \& y$-axes lie in the plane of the Lamina and $z$-axis is perpendicular to its plane and passes through the point of intersecting of $x$ and $y$ axes.
.Theorem of parallel axes. It states that the moment of inertia of a rigid body about any axis is equal to moment of inertia of the body about a parallel axis through its center of mass plus the product of mass of the body and the square of the perpendicular distance between the axes.

Mathematically: $I=I_{c}+M h^{2}$, where $I_{c}$ is moment of inertia of the body about an axis through its centre of mass and $h$ is the perpendicular distance between the two axes.
.Moment of inertia of a few bodies of regular shapes:
i. M.I. of a rod about an axis through its c.m. and perpendicular to rod, $I=\frac{1}{12} M L^{2}$
ii. M.I. of a circular ring about an axis through its centre and perpendicular to its plane, $I=M R^{2}$
iii. M.I. of a circular disc about an axis through its centre and perpendicular to its plane, $I=\frac{1}{2} M R^{2}$
iv. M.I. of a right circular solid cylinder about its symmetry axis, $I=$ $\frac{1}{2} M R^{2}$
v. M.I. of a right circular hollow cylinder about its axis $=M R^{2}$
vi. M.I. of a solid sphere about its diameter, $I=\frac{2}{5} M R^{2}$
vii. M.I. of spherical shell about its diameter, $I=\frac{2}{3} M R^{2}$
.Moment of inertia and angular momentum. The moment of inertia of a rigid body about an axis is numerically equal to the angular momentum of the rigid body, when rotating with unit angular velocity about that axis.

Mathematically: $K . E$ of rotation $=\frac{1}{2} I \omega^{2}$
.Moment of inertia and kinetic energy of rotation. The moment of inertia of a rigid body about an axis of rotation is numerically equal to twice the kinetic energy of rotation of the body, when rotation with unit angular velocity about that axis.

Mathematically:K.E. of rotation $=\frac{1}{2} I \omega^{2}$
.Moment of inertia and torque. The moment of inertia of a rigid body about an axis of rotation is numerically equal to the external torque required to produce a unit angular acceleration in the body BOUT THE GIVEN AXIS.

MATHEMATICALLY: $\tau=I a$
.Law of conservation of angular momentum. If no external torque acts on a system, the total angular momentum of the system remains unchanged.

Mathematically:
$I \omega=$ constant vector, i.e., in magnitude, $I_{1} \omega_{1}=I_{2} \omega_{2}$, provides no external torque acts on the system.

For translational equilibrium of a rigid body, $\vec{F}=\sum_{i} F_{i}=0$
For rotational equilibrium of a rigid body, $\quad \vec{\tau}=\sum_{i} \vec{\tau}_{i}=0$
1.The following table gives a summary of the analogy between various quantities describing linear motion and rotational motion.

1. Distance/displacement (s)
2. Linear velocity, $\vartheta=\frac{d s}{d t}$
3. 

Linear acceleration,
$\alpha=\frac{d v}{d t}=\frac{d^{2} r}{d r^{2}}$

Mass (m)
4.

Linear momentum, $\boldsymbol{p}=\boldsymbol{m} \boldsymbol{v}$
5.

Force, $\boldsymbol{F}=\boldsymbol{m} \boldsymbol{a}$
6.

Also, force $F=\frac{d p}{d t}$
7.
8. Translational KE, $K_{T}=\frac{1}{2} \boldsymbol{m} v^{2}$
9. Work done, $W=F s$

Power, $P=F \boldsymbol{v}$

1. Angle or angular displacement ( $\boldsymbol{\theta}$ )
2. Angular velocity, $\omega=\frac{d \theta}{d t}$
3. Angular acceleration $=\alpha=$ $\frac{d \omega}{d t}=\frac{d^{2} \theta}{d r^{2}}$

Moment of inertia (I)
4.

Angular momentum, $L=I \omega$
5.

Torque, $\tau=I a$
6.

Also, torque, $\tau=\frac{d L}{d t}$

$$
7 .
$$

8. 

Work done, $\boldsymbol{W}=\boldsymbol{\tau} \boldsymbol{\theta}$
Rotational KE, $K_{R}=\frac{1}{2} I \omega^{2}$
9.
11.
12.

Linear momentum of a system is conserved when no external

Equation of translator motion
i. $v=u+a t$
ii. $\quad s=u t+\frac{1}{2} a t^{2}$
iii. $v^{2}-u^{2}=$

2as, where the symbol: have their usual meaning.
10. force acts on the system.
11.
12.

Angular momentum of a system is conserved when no external torque acts on the system

Equations of rotational motion
i. $\quad \omega_{2}=\omega_{1}+a t$
ii. $\quad \theta=\omega_{1} t+\frac{1}{2} a t^{2}$
iii. $\quad \omega_{2}^{2}-\omega_{1}^{2}=2 a \theta$, where the symbols have their usual meaning.


## 1 Marks Questions

1. If one of the particles is heavier than the other, to which will their centre of mass shift?

Answer:- The centre of mass will shift closer to the heavier particle.
2. Can centre of mass of a body coincide with geometrical centre of the body? Answer:- Yes, when the body has a uniform mass density.
3.Which physical quantity is represented by a product of the moment of inertia and the angular velocity?
Answer:- Product of I and $\omega$ represents angular momentum( $L=I \omega$ ).
4.What is the angle between $\vec{A}$ and $\vec{B}$, if $\vec{A}$ and $\vec{B}$ denote the adjacent sides of a parallelogram drawn from a point and the area of parallelogram is $\frac{1}{2} A B$. Answer:- Area of parallelogram $=|\vec{A} X \vec{B}|=A B \sin \theta=\frac{1}{2} A B$. (Given)

$$
\sin \theta=\frac{1}{2}=\sin 30^{\circ} \text { or } \theta=30^{\circ}
$$

5. Which component of linear momentum does not contribute to angular momentum?

Answer:- The radial component of linear momentum makes no contribution to angular momentum.
6. A disc of metal is melted and recast in the form of solid sphere. What will happen to the moment of inertia about a vertical axis passing through the centre ?
Answer:- Moment of inertia will decrease, because $I_{d}=\frac{1}{2} \mathrm{mr}^{2}$ and $\mathrm{I}_{\mathrm{s}}=\frac{2}{5} \mathrm{mr}^{2}$, the radius of sphere formed on recasting the disc will also decrease.
7. What is rotational analogue of mass of body?

Answer:- Rotational analogue of mass of a body is moment of inertia of the body.

## 8. What are factors on which moment of inertia depend upon?

Answer:- Moment of inertia of a body depends on position and orientation of the axis of rotation. It also depends on shape, size of the body and also on the distribution of mass of the body about the given axis.

## 9. Is radius of gyration of a body constant quantity?

Answer:- No, radius of gyration of a body depends on axis of rotation and also on distribution of mass of the body about the axis.
10. Is the angular momentum of a system always conserved? If no, under what condition is it conserved?

Answer:- No, angular momentum of a system is not always conserved. It is conserved only when no external torque acts on the system.

## 2 Marks Questions

## 1. Why is the handle of a screw made wide?

Answerwer:- Turning moment of a force $=$ force $\times$ distance( $r$ ) from the axis of rotation. To produce a given turning moment, force required is smaller, when $r$ is large. That's what happens when handle of a screw is made wide.
2. Can a body in translatory motion have angular momentum? Explain. Answer:- Yes, a body in translatory motion shall have angular momentum, the fixed point about which angular momentum is taken lies on the line of motion of the body. This follows from |L|=rpsin $\theta$.
$L=0$, only when $\theta=0^{\circ}$ or $\Theta=180^{\circ}$.
3. A person is sitting in the compartment of a train moving with uniform velocity on a smooth track. How will the velocity of centre of mass of compartment change if the person begins to run in the compartment?

Answer:- We know that velocity of centre of mass of a system changes only when an external force acts on it. The person and the compartment form one system on which no external force is applied when the person begins to run. Therefore, there will be no change in velocity of centre of mass of the compartment.
4. A particle performs uniform circular motion with an angular momentum L. If the frequency of particle's motion is doubled and its K.E is halved, what happens to the angular momentum?

Answer:- $L=m$ vr and $v=r \omega=r(2 \pi n)$

$$
r=\frac{v}{2 \pi n} \quad \therefore \quad L=m v\left(\frac{v}{2 \pi n}\right)=\frac{m v^{2}}{2 \pi n}
$$

As,

$$
\mathrm{K} . \mathrm{E}=\frac{1}{2} \mathrm{mv}^{2} \text {, therefore, } \mathrm{L}=\frac{\mathrm{K} . \mathrm{E}}{\pi \mathrm{n}}
$$

When K.E. is halved and frequency $(n)$ is doubled, $L=\frac{K \cdot E \prime}{\pi n^{\prime}}=\frac{\mathrm{K} \cdot \mathrm{E} / 2}{\pi(2 \mathrm{n})}=\frac{\mathrm{K} \cdot \mathrm{E}}{4 \pi \mathrm{n}}=\frac{\mathrm{L}}{4}$ i.e. angular momentum becomes one fourth.
5. An isolated particle of mass $m$ is moving in a horizontal plane $(x-y)$, along the $x$-axis at a certain height above the ground. It explodes suddenly into two fragments of masses m/4 and 3 m/4. An instant later, the smaller fragments is at $y=+15 \mathbf{c m}$. What is the position of larger fragment at this instant?
Answer:- As isolated particle is moving along x-axis at a certain height above the ground, there is no motion along y-axis. Further, the explosion is under internal forces only. Therefore, centre of mass remains stationary along y-axis after collision. Let the co-ordinates of centre of mass be ( $\mathrm{x}_{\mathrm{cm}}, 0$ ).
Now, $\quad y_{c m}=\frac{m_{1} y_{1}+m_{2} y_{2}}{m_{1}+m_{2}}=0 \quad \therefore \quad m_{1} y_{1}+m_{2} y_{2}=0$

Or

$$
\mathrm{y}_{2}=\frac{-\mathrm{m}_{1} \mathrm{y}_{1}}{\mathrm{~m}_{2}}=\frac{-\mathrm{m} / 4}{3 \mathrm{~m} / 4} \times 15=-5 \mathrm{~cm}
$$

So, larger fragment will be at $y=-5$; along $x$-axis.
6. Why there are two propellers in a helicopter?

Answerwer:- If there were only one propeller in a helicopter then, due to conservation of angular momentum, the helicopter itself would have turned in the opposite direction.
7. A solid wooden sphere rolls down two different inclined planes of the same height but of different inclinations. (a) Will it reach the bottom with same
speed in each case? (b) Will it take longer to roll down one inclined plane than other ? Explain.

Answer:- (a) Yes, because at the bottom depends only on height and not on slope.
(b) Yes, greater the inclination( $\theta$ ), smaller will be time of decent, as $\mathrm{t} \propto$ $1 / \sin \theta$.
8. There is a stick half of which is wooden and half is of steel. It is pivoted at the wooden end and a force is applied at the steel end at right angles to its length. Next, it is pivoted at the steel end and the same force is applied at the wooden end. In which case is angular acceleration more and why?

Answer:- We know that torque, $\tau=$ Force $\times$ Distance $=\mathrm{I} \alpha=$ constant

$$
\therefore \alpha=\frac{\tau}{1} \quad \text { i.e } \alpha \propto \frac{1}{1}
$$

Angular acc. ( $\alpha$ ) will be more, when I is small, for which lighter material(wood) should at larger distance from the axis of rotation I.e. when stick is pivoted at the steel end.
9. Using expressions for power in rotational motion, derive the relation $=\mathrm{I} \alpha$, where letters have their usual meaning.
Answer:- We know that power in rotational motion, $P=\tau \omega$
and K.E. of motion, $E=\frac{1}{2} \mathrm{I} \omega^{2}$
As power= time rate of doing work in rotational motion, and work is stored in the body in the form of K.E.

$$
\begin{aligned}
\therefore \quad \mathrm{P}=\frac{\mathrm{d}}{\mathrm{dt}} & (\text { K.E. of rotation }) \\
& =\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2} \mathrm{I} \omega\right)=\frac{1}{2} \mathrm{I} \times 2 \omega\left(\frac{\mathrm{~d} \omega}{\mathrm{dt}}\right) \\
\mathrm{P} & =\mathrm{I} \omega \alpha
\end{aligned}
$$

Using (i), $P=\tau \omega=I \omega \alpha$ or $\tau=I \alpha$, which is the required relation.
10. Calculate radius of gyration of a cylindrical rod of mass $m$ and length $L$ about an axis of rotation perpendicular to its length and passing through the centre.
Answer:- $\mathrm{K}=$ ? , mass $=\mathrm{m}$, length $=\mathrm{L}$
Moment of inertia of the rod about an axis perpendicular to its length and passing through the centre is

$$
\begin{aligned}
& \mathrm{I}=\frac{\mathrm{mL}^{2}}{12} \\
& \text { Also, } \quad \mathrm{I}=\mathrm{mK}^{2} \quad \therefore \mathrm{mK}^{2}=\frac{\mathrm{mL}^{2}}{12} \quad \text { or } \quad \mathrm{K}=\frac{\mathrm{L}}{\sqrt{12}}=\frac{\mathrm{L}}{2 \sqrt{3}} .
\end{aligned}
$$

## 3 Marks Questions

1. Explain that torque is only due to transverse component of force. Radial component has nothing to do with torque.
2. Show that centre of mass of an isolated system moves with a uniform velocity along a straight line path.
3. If angular momentum is conserved in a system whose moment of inertia is decreased, will its rotational kinetic energy be also conserved ? Explain. Ans:- Here, L = I $\omega=$ constant
K.E. of rotation, $K=\frac{1}{2} I \omega^{2}$

$$
\mathrm{K}=\frac{1}{21} \mathrm{I}^{2} \omega^{2}=\frac{\mathrm{L}^{2}}{21}
$$

As L is constant, $. \therefore \mathrm{K} \propto 1 / \mathrm{I}$
When moment of inertia(I) decreases, K.E. of rotation(K) increases. Thus K.E. of rotation is not conserved.
4. How will you distinguish between a hard boiled egg and a raw egg by spinning each on a table top?
Ans:- To distinguish between a hard boiled egg and a raw egg, we spin each on a table top. The egg which spins at a slower rate shall be raw. This is because in a raw egg, liquid matter inside tries to get away from its axis of rotation. Therefore, its moment of inertia I increases. As $\tau=\mathrm{I} \alpha=$ constant, therefore, $\alpha$ decreases i.e. raw egg will spin with smaller angular acceleration. The reverse is true for a hard boiled egg which will rotate more or less like a rigid body.
5.Equal torques are applied on a cylindrical and a hollow sphere. Both have same mass and radius. The cylinder rotates about its axis and the sphere rotates about one of its diameters. Which will acquire greater speed? Explain.
6.Locate the centre of mass of uniform triangular lamina and a uniform cone.
7. A thin wheel can stay upright on its rim for a considerable length when rolled with a considerable velocity, while it falls from its upright position at the slightest disturbance when stationary. Give reason. Answer:- When the wheel is rolling upright, it has angular momentum in the horizontal direction i.e., along the axis of the wheel. Because the angular momentum is to remain conserved, the wheel does not fall from its upright position because that would change the direction of angular momentum. The wheel falls only when it loses its angular velocity due to friction.
8. Why is the speed of whirl wind in a tornado so high? Answer:- In a whirl wind, the air from nearby region gets concentrated in a small space thereby decreasing the value of moment of inertia considerably. Since, I $\omega=$ constant, due to decrease in moment of inertia, the angular speed becomes quite high.
9. Explain the physical significance of moment of inertia and radius of gyration.
10. Obtain expression for K.E. of rolling motion.

## 5 Marks Questions

1. Define centre of mass. Obtain an expression for perpendicular of centre of mass of two particle system and generalise it for particle system.
2. Find expression for linear acceleration of a cylinder rolling down on a inclined plane.

A ring, a disc and a sphere all of them have same radius and same mass roll down
on inclined plane from the same heights. Which of these reaches the bottom (i) earliest (ii) latest ?
3. (i) Name the physical quantity corresponding to inertia in rotational motion. How is it calculated? Give its units.
(ii)Find expression for kinetic energy of a body.
4. State and prove the law of conservation of angular momentum. Give one illustration to explain it.
5. State parallel and perpendicular axis theorem.

Define an expression for moment of inertia of a disc $R$, mass $M$ about an axis along its diameter.

## TYPICAL PROBLEMS

1. A uniform disc of radius $R$ is put over another uniform disc of radius $2 R$ of the same thickness and density. The peripheries of the two discs touch each other. Locate the centre of mass of the system.

## Ans:-

Let the centre of the bigger disc be the origin.
$2 R=$ Radius of bigger disc
$R=$ Radius of smaller disc
$m_{1}=\pi R^{2} \times T \times \rho$
$m_{2}=\pi(2 R)^{2} \times T \times \rho$, where $\mathrm{T}=$ Thickness of the two discs

$\rho=$ Density of the two discs
$\therefore$ The position of the centre of mass

$$
\begin{gathered}
=\left(\frac{m_{1} x_{1}+m_{2} x_{2}}{m_{1}+m_{2}}, \frac{m_{1} y_{1}+m_{2} y_{2}}{m_{1}+m_{2}}\right) \\
x_{1}=R \quad y_{1}=0 \\
x_{2}=0 \quad y_{2}=0 \\
\\
\left(\frac{\pi R^{2} T \rho R+0}{\pi R^{2} T \rho+\pi(2 R)^{2} T \rho}, \frac{0}{m^{1}+m^{2}}\right)=\left(\frac{\pi R^{2} T \rho R}{5 \pi R^{2} T \rho}, 0\right)=\left(\frac{R}{5}, 0\right)
\end{gathered}
$$

At R/5 from the centre of bigger disc towards the centre of smaller disc.
2. Two blocks of masses 10 kg and 20 kg are placed on the x-axis. The first mass is moved on the axis by a distance of 2 cm . By what distance should the second mass be moved to keep the position of centre of mass unchanged?

Ans:- Two masses $m_{1}$ and $m_{2}$ are placed on the X-axis

$$
m_{1}=10 \mathrm{~kg} \quad, \quad m_{2}=20 \mathrm{~kg}
$$

The first mass is displaced by a distance of 2 cm

$$
\begin{gathered}
\therefore \overline{X_{c m}}=\frac{m_{1} x_{1}+m_{2} x_{2}}{m_{1}+m_{2}}=\frac{10 \times 2+20 x_{2}}{30} \\
\quad \Rightarrow 0=\frac{20+20 x_{2}}{30} \\
\quad \Rightarrow 20+20 x_{2}=0 \\
\quad \Rightarrow 20=-20 x_{2} \\
\quad \Rightarrow x_{2}=-1 c m
\end{gathered}
$$

$\therefore$ The 2nd mass should be displaced by a distance 1 cm towards left so as to kept the position of centre of mass unchanged.
3. A simple of length $l$ is pulled aside to make an angle $\theta$ with the vertical.

Find the magnitude of the torque of the weight $w$ of the bob about the point of suspension. When is the torque zero ?


Ans:- A simple of pendulum of length I is suspended from a rigid support.
A bob of weight $W$ is hanging on the other point.
When the bob is at an angle $\theta$ with the vertical,
then total torque acting on the point of suspension $=\mathrm{i}=\mathrm{F} \times \mathrm{r}$
$\Rightarrow \mathrm{Wr} \sin \theta=\mathrm{WI} \sin \theta$
At the lowest point of suspension the torque will be zero as the force acting on the body passes through the point of suspension.
4. A square plate of mass 120 g and edge 5.0 cm rotates about one of edges. If it has a uniform angular acceleration of $0.2 \mathrm{rad} / \mathrm{s}^{2}$, what torque acts on the plate ?

Ans:- A square plate of mass 120 gm and edge 5 cm rotates about one of the edge. Let take a small area of the square of width $d x$ and length a which is at a distance $x$ from the axis of
rotation.
Therefore mass of that small area
$\mathrm{m} / \mathrm{a}^{2} \times \mathrm{adx}(\mathrm{m}=\mathrm{mass}$ of the square ; $\mathrm{a}=$ side of the plate)

$\left.I=\int_{0}^{a}\left(m / a^{2}\right) \times a x^{2} d x=(m / a)\left(x^{3} / 3\right)\right]_{0}^{\frac{a}{0}}$

$$
=m a^{2} / 3
$$

Therefore torque produced $=1 \times \alpha=\left(m a^{2} / 3\right) \times \alpha$

$$
\begin{aligned}
& =\left\{\left(120 \times 10^{-3} \times 5^{2} \times 10^{-4}\right) / 3\right\} 0.2 \\
& =0.2 \times 10^{-4}=2 \times 10^{-5} \mathrm{~N}-\mathrm{m} .
\end{aligned}
$$

5. A wheel of moment of inertia $0.10 \mathrm{~kg}-\mathrm{m}^{2}$ is rotating about a shaft at an angular speed of 160 rev/minute. A second wheel is set into rotation at 300 rev/minute and is coupled to the same shaft so that both the wheels finally rotate with a common angular speed of 200 rev/minute. Find the moment of
inertia of the second wheel.
Ans:- Wheel (1) has
$I_{1}=0.10 \mathrm{~kg}-\mathrm{m}^{2}, \omega_{1}=160 \mathrm{rev} / \mathrm{min}$
Wheel (2) has
$I_{2}=$ ? ; $\omega_{2}=300 \mathrm{rev} / \mathrm{min}$
Given that after they are coupled, $\omega=200 \mathrm{rev} / \mathrm{min}$
Therefore if we take the two wheels to bean isolated system
Total external torque $=0$
Therefore, $I_{1} \omega_{1}+I_{1} \omega_{2}=\left(I_{1}+I_{1}\right) \omega$

$\Rightarrow 0.10 \times 160+I_{2} \times 300=\left(0.10+I_{2}\right) \times 200$
$\Rightarrow 5 I_{2}=1-0.8$
$\Rightarrow I_{2}=0.04 \mathrm{~kg}-\mathrm{m}^{2}$.
