Relations and Functions

Flashback from Class XI Maths

1. What is Cartesian Sets?

Given two non-empty sets A and B. The Cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B, i.e.,

$A \times B = \left\{ {\left({a,b} \right):a \in A,b \in B} \right\}$

If either A or B is the null set, then $A \times B$ will also be empty set, i.e., $A \times B$

2. What is relations?

A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$.

It *"maps"* elements of one set to another set. The subset is derived by describing a relationship between the first element and the second element of the ordered pair $\left({A \times B} \right)$.

Domain: The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

Range: the set of all the ending points is called the *range*

3. What is Function

A function is a "well-behaved" relation

A function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

For a relation to be a function, there must be only and exactly one y that corresponds to a given x

If f is a function from A to B and $\left({a,{\rm{ }}b} \right) \in f$, then $f\left(a \right) = b$, where b is called the image of a under f and a is called the preimage of b under f.

4. Algebra of Real Function

Real Value Function: A function which has all real number or subset of the real number as it domain

Real Valued Function: A function which has all real number or subset of the real number as it range

For functions $f:{\rm{ }}X - > {\bf{R}}$ and $g:{\rm{ }}X - > {\bf{R}}$, we have

Addition

$\left({f + g} \right)\left(x \right) = f\left(x \right) + g\left(x \right),x \in X$

Substraction

$\left({f - g} \right)\left(x \right) = f\left(x \right)-g\left(x \right),x \in X$

Multiplication

$\left({f.g} \right)\left(x \right) = f\left(x \right).g\left(x \right),x \in X$

Multiplication by real number $\left({kf} \right)\left(x \right) = k{\rm{ }}f\left(x \right),x \in X$, where k is a real number.

Division

$\frac{f}{g}\left(x \right) = \frac{{f(x)}}{{g(x)}}$

$x \in X$ and $g\left(x \right) \ne 0$

relation from A to A" we instead say that R is a "relation on A".

5. Type Of Relation's

Empty Relation

A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., R = φ □ A × A.

Universal Relation

A relation R in a set A is called universal relation if all elements of A is related to every element of A i.e R= A X A.

Reflexive Relation

A relation in a set A is called reflexive relation if (a,a) □ R for every element a □ A.

Example:.

Let A = {1, 2, 3, 4,5,6,7,8,9,10} and define R = {(a, b) | a divides b}

We saw that R was reflexive since every number divides itself

Let A = {1, 2, 3, 4,5,6,7,8,9,10} and define R ={(1,1),(2,2),(2,3),(3,2),(4,4)}

We saw that R is not reflexive since every number is not present in R

Symmetric Relation

A relation in a set A is called if (a,b) □ R the (b,a) □ R for all a,b □ A

Example

Let A = {1, 2, 3, 4} and R = {(1, 2), (2,1),(3,3),(4,4)}

This relation is symmetric. It satisfies the above criterion.

Important Note: symmetry is a different kind of requirement than reflexivity. Reflexivity requires that certain pairs must be in R, namely all pairs of the form (a, a) for every element in A. However symmetry only requires that if a pair (a, b) is in R, then the pair (b, a) must also be in R. But it is not required that pairs of the form (a, b) are in R unless the pair (b, a) is in R. Simply stated, you must have both pairs or neither

Transitive Relation

A relation R on a set A is called transitive if whenever (a, b) is in R and (b, c) is in R, then (a, c) is in R.

Example :

Let A = Z and define R = {(a, b) | a > b}.

R is transitive because if a > b and b > c then a > c.

Important Note:

Note that transitivity, like symmetry, is possessed by a relation unless the stated condition is violated. So unless you can find pairs (a, b) and (b, c) which are in R while (a, c) is not, then the relation is transitive. In particular, the empty relation is always transitive because it has no pairs to violate the condition

Equivalence Relation

A relation R on a Set A is called equivalence relation if R is reflexive,Symmetric and transistive

Example:

Let A = Z and define R = {(a, b) | 3 divides a - b}.

It is reflexive as (a-a) will always be divided by 3

It is symmetric as if (a-b) will be divided by 3 ,the (b-a) will also divided by 3

it is transitive as if (a,b) □ R and (b,) □ R which means a-b and b-c are divided by 3, now a-c=(a-b)+(b-c) ,so (a,c) □ R

Question:

Determine if the below relation is an equivalence Relation

A={1,2,3,4,5,.....,14)

R is the relation on set A defined as

R={(a,b)| (3a-b)=0}

Solution :

From the defination , R would contain following elements

R={(1,3),(2,6),(3,9),(4,12)}

R is not reflexive as we dont have (1,1),(2,2) like that in the Relation

R is not symmetric also as we dont (3,1) for (1,3) in the relation

R is not transitive also as dont have (1,9) for (1,3), (3,9) in R

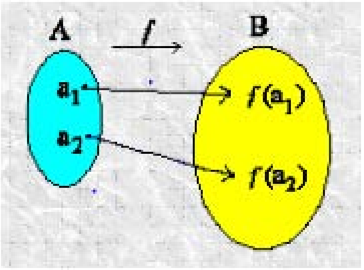
6 . Type Of Functions:

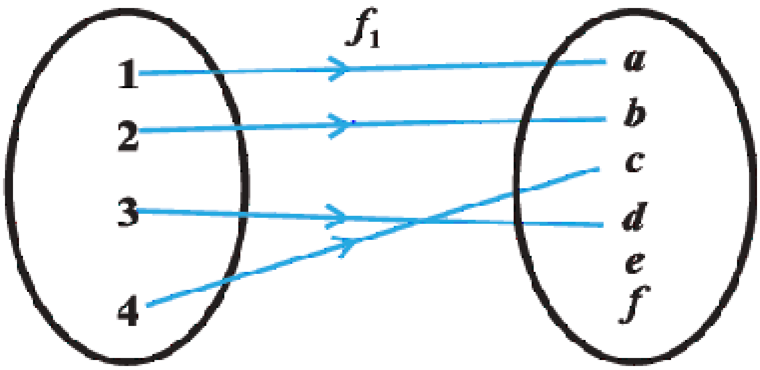
One to One Function or Injective :

Let f: A -> B, a function from a set A to a set B, f is called a one-to-one function or injection, if, and only if, for all elements a_1 and a_2 in A,
if $f(a_1) = f(a_2)$, then $a_1 = a_2$

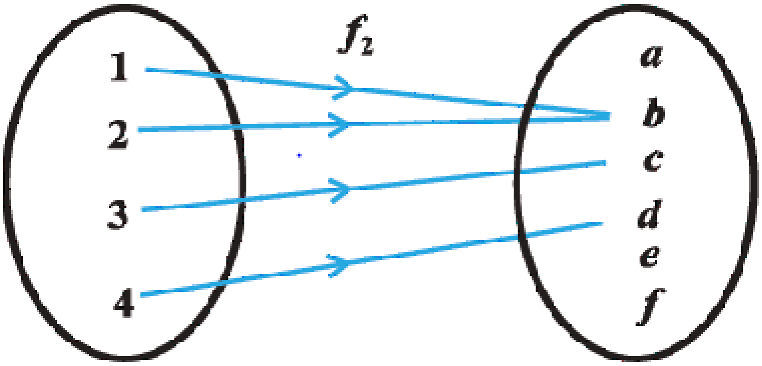
Equivalently,

if $a_1 \neq a_1$, then $f(a_1) \neq f(a_2)$.





A function is not one on one if this condition is met, then it is called **many one function**
$f(a_1) = f(a_2)$, then $a_1 \neq a_2$



How to Prove a Function for One to One:

1) We have to prove that

if $f(a_1) = f(a_2)$, then $a_1 = a_2$

2) Let us take example as

f : R-> R

$f(x) = 3x-2$ for $x \in R$

3) $f(a_1) = 3a_1-2$

$f(a_2)=3a_2-2$

Now $f(a_1) = f(a_2)$,

$3a_1-2=3a_2-2$

or $a_1 = a_2$

So our first point is proved,so this is one to one function

How to prove a function for Many one 's (Not one to one)

1) We have to prove that

if $f(a_1) = f(a_2)$, then $a_1 \neq a_2$

2) We can prove this by giving counter example

f : R-> R

$f(x) = x^2$ for $x \in R$

Now for x=2 f(2)=4

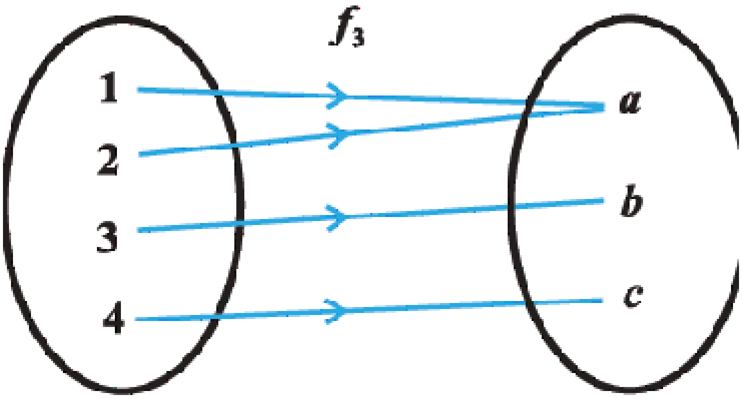
for x=-2 ,f(-2)=4

Now f(2)=f(-2) Hence Many one

Onto (Surjective) Function:

A function f: A-> B is said to be onto(surjective) if every element of B is the image of some element of A under f, i,e

for every $y \in B$, there exists a element x in A where f(x)=y



How to Prove a Function for Onto (Surjectivity):

1) for every y □ B, there exists a element x in A where f(x)=y
2) Let us take example as

f : R-> R

f(x) =3x-2 for x □ R

3) Let y □ R, then we first need to prove that we can find x □ R such that f(x)=y

If such real number x exists then y=3x-2

or x=(y+2)/3

Now sum and division of real number is a real number,So x □ R

it follows

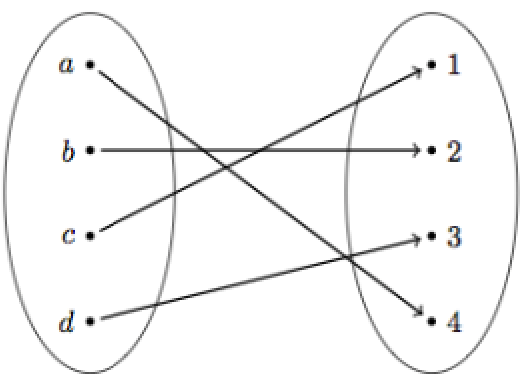
f((y+2)/3)= 3(y+2)/3 -2=y

hence f is onto function

4) if we have to prove that function is not onto,then easiast way would be to take counter examples

Bijective (One on One and onto) Function

A function is said to be Bijective if it is both one on one and onto function.



How to Prove a Function for Bijectivity

To prove a function is bijective, you need to prove that it is injective and also surjective.

"Injective" means no two elements in the domain of the function gets mapped to the same image.The method has been already described above.

"Surjective" means that any element in the range of the function is hit by the function.The method has been

7. Composition of Functions:

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function gof : A -> C given by

gof=g(f(x) for all x □ A

Example

$f(x) =(x+3)$

$g(x) =x^2$

$gof=g(f(x))=g(x+3)=(x+3)^2$

Similarly

$fog=f(g(x))=f(x^2)=x^2 +3$

In this case

$fog \neq gof$

8. Invertible Function:

If the Function f : A-> B is both one to one and onto i.e bijective ,then we can find a function g: B-> A such that

g(y)=x when y=f(x). It is denoted as f^{-1}. The function f(x) is called invertible function

Another defination of Invertible function A Function f : A-> B is invertible if we can find a function g: B- > A such that

fog=y gof=x

Example

A set A is defined as A={a,b,c}

Let f: A-> A be the function defined as are

1) f={(a,a),(b,b),(c,c)}

2) f={(a,b),(b,a),(c,c)}

3) f={(a,c),(b,c),(c,a)}

Find if all these function defined are invertible

Solutions 1) The neccesary condition for invertibleness is one on one and onto

This function is clearly one on one and onto,so it is invertible

2) This function is clearly one on one and onto,so it is invertible

3) This function is not one on one and neither onto,so it is not invertible

Chapter-01

Relation and Function

TYPES OF RELATIONS:

- A relation R in a set A is called reflexive if (a, a) ∈ R for every a ∈ A.
- A relation R in a set A is called symmetric if (a1, a2) ∈ R implies that (a2, a1) ∈ R, for all a1,
- A relation R in a set A is called transitive if (a1, a2) ∈ R, and (a2, a3) ∈ R together imply that (a1
- all a1, a2, a3 ∈ A.

EQUIVALENCE RELATION

- A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

  Equivalence Classes

- Every arbitrary equivalence relation R in a set X divides X into mutually disjoint subsets (Ai) called partitions or subdivisions of X satisfying the following conditions:
- All elements of Ai are related to each other for all i
- No element of Ai is related to any element of Aj whenever i ≠ j
- $Ai \cup Ai\ = X\ and\ Ai \cap Ai\ = \Phi,\ i \neq j$. These subsets $((A_i))$ are called equivalence classes.
- For an equivalence relation in a set X, the equivalence class containing a ∈ X, denoted by [a], is the subset of X containing all elements b related to a.

  ****Function: Arelation f: A** $\longrightarrow$ B is said to be a function if every clement of A is correlated to a

  Unique element in B.

  ***Aisdomain**

  *** Biscodomain**