Chapter 4: Determinants

Determinant

Every square matrix A is associated with a number, called its determinant and it is denoted by det (A) or |A| .

Only square matrices have determinants. The matrices which are not square do not have determinants

(i) First Order Determinant

If A = [a], then det (A) = |A| = a

(ii) Second Order Determinant

$|A| = a_{11}a_{22} - a_{21}a_{12}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

(iii) Third Order Determinant

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{or } |A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Evaluation of Determinant of Square Matrix of Order 3 by Sarrus Rule

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then determinant can be formed by enlarging the matrix by adjoining the first two columns on the right and draw lines as show below parallel and perpendicular to the diagonal.

The value of the determinant, thus will be the sum of the product of element. in line parallel to the diagonal minus the sum of the product of elements in line perpendicular to the line segment. Thus,

$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$.

Note This method doesn't work for determinants of order greater than 3.

Properties of Determinants

(i) The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows e.g., |A'| = |A|

(ii) If $A = [a_{ij}]_{n \times n}$, n > 1 and B be the matrix obtained from A by interchanging two of its rows or columns, then

det (B) = – det (A)

(iii) If two rows (or columns) of a square matrix A are proportional, then |A| = O.

(iv) |B| = k |A| ,where B is the matrix obtained from A, by multiplying one row (or column) of A by k.

(v) |kA| = kn|A|, where A is a matrix of order n x n.

(vi) If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants, e.g.,

$$\begin{vmatrix} a_1+a_2 & b & c \\ p_1+p_2 & q & r \\ u_1+u_2 & v & \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & \end{vmatrix}$$

(vii) If the same multiple of the elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged, e.g.,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{31} & a_{12}+ka_{32} & a_{13}+ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(viii) If each element of a row (or column) of a determinant is zero, then its value is zero.

(ix) If any two rows (columns) of a determinant are identical, then its value is zero.

(x) If each element of row (column) of a determinant is expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

Important Results on Determinants

(i) |AB| = |A||B| , where A and B are square matrices of the same order.

(ii) $|A^n| = |A|^n$

(iii) If A, B and C are square matrices of the same order such that ith column (or row) of A is the sum of i th columns (or rows) of B and C and all other columns (or rows) of A, Band C are identical, then |A| =|B| + |C|

(iv) $|I_n|$ = 1,where I_n is identity matrix of order n

(v) $|O_n|$ = 0, where O_n is a zero matrix of order n

(vi) If Δ(x) be a 3rd order determinant having polynomials as its elements.

(a) If Δ(a) has 2 rows (or columns) proportional, then (x – a) is a factor of Δ(x).

(b) If Δ(a) has 3 rows (or columns) proportional, then $(x – a)^2$ is a factor of Δ(x). ,

(vii) A square matrix A is non-singular, if |A| ≠ 0 and singular, if |A| =0.

(viii) Determinant of a skew-symmetric matrix of odd order is zero and of even order is a nonzero perfect square.

(ix) In general, |B + C| ≠ |B| + |C|

(x) Determinant of a diagonal matrix = Product of its diagonal elements

(xi) Determinant of a triangular matrix = Product of its diagonal elements

(xii) A square matrix of order n, is non-singular, if its rank r = n i.e., if |A| ≠ 0, then rank (A) = n

(xiii) If $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ a & b & c \end{vmatrix}$, then

(a) $$\sum_{x=1}^{n} \Delta(x) = \begin{vmatrix} \sum_{x=1}^{n} f_1(x) & \sum_{x=1}^{n} f_2(x) & \sum_{x=1}^{n} f_3(x) \\ \sum_{x=1}^{n} g_1(x) & \sum_{x=1}^{n} g_2(x) & \sum_{x=1}^{n} g_3(x) \\ a & b & c \end{vmatrix}$$

(b) $$\prod_{x=1}^{n} \Delta(x) = \begin{vmatrix} \prod_{x=1}^{n} f_1(x) & \prod_{x=1}^{n} f_2(x) & \prod_{x=1}^{n} f_3(x) \\ \prod_{x=1}^{n} g_1(x) & \prod_{x=1}^{n} g_2(x) & \prod_{x=1}^{n} g_3(x) \\ a & b & c \end{vmatrix}$$

(xiv) If A is a non-singular matrix, then $|A^{-1}| = 1 / |A| = |A|^{-1}$

(xv) Determinant of a orthogonal matrix = 1 or – 1.

(xvi) Determinant of a hermitian matrix is purely real .

(xvii) If A and B are non-zero matrices and AB = 0, then it implies |A| = 0 and |B| = 0.

Minors and Cofactors

$$\text{If } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

then the minor M_{ij} of the element a_{ij} is the determinant obtained by deleting the i row and jth column.

$$i.e., \quad M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \text{minor } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\text{and} \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The cofactor of the element a_{ij} is C_{ij} = (- 1)i + j M_{ij}

Adjoint of a Matrix :-

Adjoint of a matrix is the transpose of the matrix of cofactors of the give matrix, i.e.,

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Properties of Minors and Cofactors

(i) The sum of the products of elements of .any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero, i.e., if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

then $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$ ans so on.

(ii) The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) is Δ

$$i.e., \text{ If } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{then } |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

(iii) In general, if $|A| = \Delta$, then $|A| = \sum_{i=1}^{n} a_{ij}\, C_{ij}$

and $(\text{adj } A) = \Delta^{n-1}$, where A is a matrix of order $n \times n$.

Differentiation of Determinant

$$\text{If } \Delta(x) = \begin{vmatrix} a(x) & b(x) & c(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & (x) \end{vmatrix}$$

$$\text{then } \frac{d\Delta}{dx} = \begin{vmatrix} a'(x) & b'(x) & c'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & (x) \end{vmatrix} + \begin{vmatrix} a(x) & b(x) & c(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & (x) \end{vmatrix} + \begin{vmatrix} a(x) & b(x) & c(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & '(x) \end{vmatrix}$$

Integration of Determinant

$$\text{If} \quad \Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{then } \int \Delta(x)\, dx = \begin{vmatrix} \int a_{11}(x)\, dx & \int a_{12}(x)\, dx & \int a_{13}(x)\, dx \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If the elements of more than one column or rows are functions of x, then the integration can be done only after evaluation/expansion of the determinant.

Solution of Linear equations by Determinant/Cramer's Rule

Case 1.

The solution of the system of simultaneous linear equations

$a_1x + b_1y = C_1$ ...(i)

$a_2x + b_2y = C_2$ ...(ii)

is given by $x = D_1 / D$, $Y = D_2 / D$

$$\text{where, } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \text{ provided that } D \neq 0.$$

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = D_1 / D$, $y = D_2 / D$

(ii) If D = 0 and Dl = D2 = 0, then the system is consistent and has infinitely many solutions.

(iii) If D = 0 and one of Dl and D2 is non-zero, then the system is inconsistent.

Case II.

Let the system of equations be

$a1x + b1y + C1z = d1$

$a2x + b2y + C2z = d2$

$a3x + b3y + C3z = d3$

Then, the solution of the system of equation is x = D1 / D, Y = D2 / D, Z = D3 / D, it is called Cramer's rule.

$$\text{where, } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

(i) If D ≠ 0, then the system of equations is consistent with unique solution.

(ii) If D = 0 and atleast one of the determinant D_1, D_2, D_3 is non-zero, then the given system is inconsistent, i.e., having no solution.

(iii) If D = 0 and $D_1 = D_2 = D_3 = 0$, then the system is consistent, with infinitely many solutions.

(iv) If D ≠ 0 and $D_1 = D_2 = D_3 = 0$, then system has only trivial solution, (x = y = z = 0).

Cayley-Hamilton Theorem

Every matrix satisfies its characteristic equation, i.e., if A be a square matrix, then |A – xl| = 0 is the characteristics equation of A. The values of x are called eigenvalues of A.

i.e., if $x_3 - 4x_2 - 5x - 7 = 0$ is characteristic equation for A, then $A_3 - 4A_2 + 5A - 7I = 0$

Properties of Characteristic Equation

(i) The sum of the eigenvalues of A is equal to its trace.

(ii) The product of the eigenvalues of A is equal to its determinant.

(iii) The eigenvalues of an orthogonal matrix are of unit modulus.

(iv) The feigen values of a unitary matrix are of unit modulus.

(v) A and A' have same eigenvalues.

(vi) The eigenvalues of a skew-hermitian matrix are either purely imaginary or zero.

(vii) If x is an eigenvalue of A, then x is the eigenvalue of A^* .

(viii) The eigenvalues of a triangular matrix are its diagonal elements.

(ix) If x is the eigenvalue of A and |A| ≠ 0, then (1 / x) is the eigenvalue of A^{-1}.

(x) If x is the eigenvalue of A and |A| ≠ 0, then |A| / x is the eigenvalue of adj (A).

(xi) If x_1, x_2,x_3, ... ,x_n are eigenvalues of A, then the eigenvalues of A^2 are $x_2{}^2$, $x_2{}^2$,..., $x_n{}^2$.

Cyclic Determinants

$$\text{(i)} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

$$\text{(ii)} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

$$\text{(iii)} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)\,[(a^2+b^2+c^2)+(ab+bc+ca)]$$

$$\text{(iv)} \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\text{(v)} \begin{vmatrix} x^2 & (x+a)^2 & (x-a)^2 \\ y^2 & (y+a)^2 & (y-a)^2 \\ z^2 & (z+a)^2 & (z-a)^2 \end{vmatrix} = -\,4a^3(x-y)(y-z)(z-x)$$

$$\text{(vi)} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b & a & c \end{vmatrix} = a^2+b^2+c^2-ab-bc-ca$$

$$= \frac{1}{2}[(b-c)^2+(c-a)^2+(a-b)^2]$$

$$\text{(vii)} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= -(a^3+b^3+c^3-3abc)$$

$$\text{(viii)} \begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = x^3\,(x+a+b+c+d)$$

Applications of Determinant in Geometry

Let three points in a plane be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

$$\text{(i) Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$= 1 / 2\,[x_1 (y_2 - y3) + x_2 (y3 - y_1) + x3 (y_1 - y_2)]$

(ii) If three points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(iii) Equation of a line passing through the points A and B is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Maximum and Minimum Value of Determinants

If $\quad |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix},$

where a_is $\quad [\alpha_1, \alpha_2, \ldots, \alpha_n]$

Then, $|A|_{max}$ when diagonal elements are

$\{\min(\alpha_1, \alpha_2, \ldots, \alpha_n)\}$

and non-diagonal elements are

$\{\max(\alpha_1, \alpha_2, \ldots, \alpha_n)\}$

Also, $|A|_{min} = -\,|A|_{max}$

Chapter-4

Determinant

- Determinant of a matrix $A=[a_{11}]_{1\times1}$ is given by $|a_{11}|=a_{11}$
- Determinant of a matrix A $\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$ is given by

$$|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}\ a_{22}\ -a_{12}\ \ a_{21}$$

- Determinant of a matrix A $\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$ is given by (expanding along (R_1)

$$|A|=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

- **For any square matrix A, the |A| satisfy following properties.**
- $|A'| = |A|$, where A' = transpose of A.
- If we interchange any two rows (or columns), then sign of determinant changes.
- If any two rows or any two columns are identical or proportional, then value of determinant is zero.
- If we multiply each element of a row or a column of a determinant by constant k, then value of determinant is multiplied by k.
- Multiplying a determinant by k means multiply elements of only one row (or one column) by k.
- *If* $A=\left[a_{ij}\right]_{3\times3}$ *, then* $|k.\ A| = k^3\,|A|$
- If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.

- If to each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows or columns are added, then value of determinant remains same.
- Area of a triangle with vertices (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Minor of an element aij of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and denoted by M_{ij}
- Cofactor of aij of given by Aij = (– 1)i+ j Mij
- Value of determinant of a matrix A is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example, $|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13}A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For example, $a_{11} A_{21} + a_{12}A_{22} + a_{13} A_{23} = 0$
- $A\,(adj\ A) = (adj\ A)\ A = |A|\ I$, where A is square matrix of order n.
- A square matrix A is said to be singular or non-singular according as $|A| = 0$ *or* $|A| \neq 0$.
- If AB = BA = I, where B is square matrix, then B is called inverse of A. Also $A^{-1} = B$ *or* $B^{-1} = A$ *and hence* $\left(A^{-1}\right)^{-1} = A$.
- A square matrix A has inverse if and only if A is non-singular.
- $A^{-1} = \frac{1}{|A|}(\text{adj } A)$
- If $a_1x + b_1y + c_1z = d_1$
- $a_2x + b_2y + c_2z = d_2$
- $a_3x + b_3y + c_3z = d_3$
- then these equations can be written as A X = B, where

- $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = X \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
- Unique solution of equation AX = B is given by $X = A^{-1} B,\ where\ |A|\ 0\ \neq 0$.
- A system of equation is consistent or inconsistent according as its solution exists or not.
- For a square matrix A in matrix equation AX = B
- $| |A| \neq 0,$ there exists unique solution
- $|A| = 0$ and (adj A) B ≠ 0, then there exists no solution
- $|A| = 0\ and\ (adj\ A)\ B = 0,$ then system may or may not be consistent.