Chapter 5: Continuity and Differentiability

Derivative

The rate of change of a quantity y with respect to another quantity x is called the derivative or differential coefficient of y with respect to x .

Differentiation of a Function

Let f(x) is a function differentiable in an interval [a, b]. That is, at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define a new function g: [a, b] → R, such that, x [a, b], g(x) = f′(x).

This new function is said to be differentiation (differential coefficient) of the function f(x) with respect to x and it is denoted by df(x) / d(x) or Df(x) or f′(x).

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Differentiation ‘from First Principle

Let f(x) is a function finitely differentiable at every point on the real number line. Then, its derivative is given by

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Standard Differentiations

1. d / d(x) $(x^n) = nx^{n-1}$, x R, n R
2. d / d(x) (k) = 0, where k is constant.
3. d / d(x) $(e^x) = e^x$
4. d / d(x) $(a^x) = a^x$ loge a > 0, a ≠ 1

5. $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$

6. $\frac{d}{dx}(\log_a x) = \frac{1}{x}(\log_a e) = \frac{1}{x \log_e a}$

7. $\frac{d}{dx}(\sin x) = \cos x$

8. $\frac{d}{dx}(\cos x) = -\sin x$

9. $\frac{d}{dx}(\tan x) = \sec^2 x, x \neq (2n+1)\frac{\pi}{2}, n \in I$

10. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x, x \neq n\pi, n \in I$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x, x \neq (2n+1)\frac{\pi}{2}, n \in I$

12. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x, x \neq n\pi, n \in I$

13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

14. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

15. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

16. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

18. $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

19. $\frac{d}{dx}(\sinh x) = \cosh x$

20. $\frac{d}{dx}(\cosh x) = \sinh x$

21. $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

Fundamental Rules for Differentiation

(i) $\frac{d}{dx}\{cf(x)\} = c\frac{d}{dx}f(x)$, where c is a constant.

(ii) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ (sum and difference rule)

(iii) $\frac{d}{dx}\{f(x)\,g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$ (product rule)

Generalization If $u_1, u_2, u_3, \dots, u_n$ be a function of x, then

$$\frac{d}{dx}(u_1 u_2 u_3 \dots u_n) = \left(\frac{du_1}{dx}\right)[u_2 u_3 \dots u_n] + u_1\left(\frac{du_2}{dx}\right)[u_3 \dots u_n] + u_1 u_2\left(\frac{du_3}{dx}\right)[u_4 u_5 \dots u_n] + \dots + [u_1 u_2 \dots u_{n-1}]\left(\frac{du_n}{dx}\right)$$

(iv) $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\{g(x)^2\}}$ (quotient rule)

(v) if d / d(x) f(x) = φ(x), then d / d(x) f(ax + b) = a φ(ax + b)

(vi) Differentiation of a constant function is zero i.e., d / d(x) (c) = 0.

Geometrically Meaning of Derivative at a Point

Geometrically derivative of a function at a point x = c is the slope of the tangent to the curve y = f(x) at the point {c, f(c)}.

Slope of tangent at P = lim $_{x \to c}$ f(x) – f(c) / x – c = {df(x) / d(x)}$_{x = c}$ or f’ (c).

Different Types of Differentiable Function

1. Differentiation of Composite Function (Chain Rule)

If f and g are differentiable functions in their domain, then fog is also differentiable and

(fog)’ (x) = f’ {g(x)} g’ (x)

More easily, if y = f(u) and u = g(x), then dy / dx = dy / du * du / dx.

If y is a function of u, u is a function of v and v is a function of x. Then, dy / dx = dy / du * du / dv * dv / dx.

2. Differentiation Using Substitution

In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below .

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$
(iv)	$\sqrt{a + x}$ and $\sqrt{a - x}$	$x = a\cos 2\theta$
(v)	$a\sin x + b\cos x$	$a = r\cos\alpha$, $b = r\sin\alpha$
(vi)	$\sqrt{x - \alpha}$ and $\sqrt{\beta - x}$	$x = \alpha\sin^2\theta + \beta\cos^2\theta$
(vii)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

3. Differentiation of Implicit Functions

If f(x, y) = 0, differentiate with respect to x and collect the terms containing dy / dx at one side and find dy / dx.

Shortcut for Implicit Functions For Implicit function, put d /dx {f(x, y)} = – ∂f / ∂x / ∂f / ∂y, where ∂f / ∂x is a partial differential of given function with respect to x and ∂f / ∂y means Partial differential of given function with respect to y.

4. Differentiation of Parametric Functions

If x = f(t), y = g(t), where t is parameter, then dy / dx = (dy / dt) / (dx / dt) = d / dt g(t) / d / dt f(t) = g’ (t) / f’ (t)

5. Differential Coefficient Using Inverse Trigonometrical Substitutions

Sometimes the given function can be deducted with the help of inverse Trigonometrical substitution and then to find the differential coefficient is very easy.

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1 - x^2})$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$ or $\cos^{-1}(1 - 2x^2)$

(iii) $2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right) \\ \tan^{-1}\left(\dfrac{2x - x^2}{1}\right) \\ \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right) \end{cases}$

(iv) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

(v) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

(vi) $3\tan^{-1}x = \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

(vii) $\cos^{-1}x + \sin^{-1}x = \pi/2$

(viii) $\tan^{-1}x + \cot^{-1}x = \pi/2$

(ix) $\sec^{-1}x + \mathrm{cosec}^{-1}x = \pi/2$

(x) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1 - y^2} \pm y\sqrt{1 - x^2}\right]$

(xi) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{(1 - x^2)(1 - y^2)}\right]$

(xii) $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left[\dfrac{x \pm y}{1 \mp xy}\right]$

Logarithmic Differentiation Function

(i) If a function is the product and quotient of functions such as y = $f_1(x)\ f_2(x)\ f_3(x)\ldots$ / $g_1(x)\ g_2(x)\ g_3(x)\ldots$, we first take algorithm and then differentiate.

(ii) If a function is in the form of exponent of a function over another function such as $[f(x)]^{g(x)}$, we first take logarithm and then differentiate.

Differentiation of a Function with Respect to Another Function

Let y = f(x) and z = g(x), then the differentiation of y with respect to z is dy / dz = dy / dx / dz / dx = f' (x) / g' (x)

Successive Differentiations

If the function y = f(x) be differentiated with respect to x, then the result dy / dx or f' (x), so obtained is a function of x (may be a constant).

Hence, dy / dx can again be differentiated with respect of x.

The differential coefficient of dy / dx with respect to x is written as d /dx (dy / dx) = d^2y / dx^2 or f' (x). Again, the differential coefficient of d^2y / dx^2 with respect to x is written as d / dx (d^2y / dx^2) = d^3y / dx^3 or f'''(x)……

Here, dy / dx, d^2y / dx^2, d^3y / dx^3,… are respectively known as first, second, third, … order differential coefficients of y with respect to x. These alternatively denoted by f' (x), f'' (x), f''' (x), … or y_1, y_2, y_3…., respectively.

Note dy / dx = (dy / dθ) / (dx / dθ) but d^2y / dx^2 ≠ (d^2y / $d\theta^2$) / (d^2x / $d\theta^2$)

Leibnitz Theorem

If u and v are functions of x such that their nth derivative exist, then

$$D^n(u\cdot v) = {}^nC_0(D^n u)v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)(D^2v) + {}^nC_3(D^{n-3}u)(D^3v) + \dots + {}^nC_r D^{n-r}u\cdot D^r v + \dots + {}^nC_n(D^n v)$$

nth Derivative of Some Functions

(i) $\dfrac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(\dfrac{n\pi}{2} + ax + b\right)$

(ii) $\dfrac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(\dfrac{n\pi}{2} + ax + b\right)$

(iii) $\dfrac{d^n}{dx^n}(ax+b)^m = \dfrac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

(iv) $\dfrac{d^n}{dx^n}[\log(ax+b)] = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

(v) $\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

(vi) $\dfrac{d^n}{dx^n}(a^x) = a^x(\log a)^n$

(vii) (a) $\dfrac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\phi)$

(b) $\dfrac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos(bx+c+n\phi)$

where, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\dfrac{b}{a}\right)$

Derivatives of Special Types of Functions

(i) If $y = f(x)^{\{f(x)\}^{\dots\infty}}$, then $\dfrac{dy}{dx} = \dfrac{y^2 f'(x)}{f(x)\{1 - y\log f(x)\}}$

(ii) If $e^{g(y)} - e^{-g(y)} = 2f(x)$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{g'(y)}\cdot\dfrac{1}{\sqrt{1+\{f(x)\}^2}}$

(iii) If $y = \sqrt{\dfrac{1+g(x)}{1-g(x)}}$, then $\dfrac{dy}{dx} = \dfrac{g'(x)}{[1-g(x)]^2}\cdot\sqrt{\dfrac{1-g(x)}{1+g(x)}}$

(iv) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots\infty}}}$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{2y-1}$

(v) If $\{f(x)\}^{g(y)} = e^{f(x)-g(y)}$, then $\dfrac{dy}{dx} = \dfrac{f'(x)\log f(x)}{g'(y)\{1+\log f(x)\}^2}$

(vi) If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then

$$\frac{dy}{dx} = \frac{g(y)}{f(x)}\cdot\frac{f'(x)}{g'(y)}\left[\frac{f(x)\log g(y) - g(y)}{g(y)\log f(x) - f(x)}\right]$$

vii) Differentiation of a Determinant

If $y = \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix}$, then

$$\frac{dy}{dx} = \begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} & \frac{dr}{dx} \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ \frac{du}{dx} & \frac{dv}{dx} & \frac{dw}{dx} \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ u & v & w \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix}$$

viii) Differentiation of Integrable Functions

If $g_1(x)$ and $g_2(x)$ are defined in [a, b], Differentiable at x [a, b] and f(t) is continuous for $g_1(a) \le f(t) \le g_2(b)$, then

$$\text{Thus, } \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\text{and } \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Partial Differentiation

The partial differential coefficient of f(x, y) with respect to x is the ordinary differential coefficient of f(x, y) when y is regarded as a constant. It is a written as $\partial f / \partial x$ or $D_x f$ or f_x.

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)dt = f[g_2(x)]\frac{d}{dx}[g_2(x)] - f[g_1(x)]\frac{d}{dx}[g_1(x)].$$

e.g., If $z = f(x, y) = x^4 + y^4 + 3xy^2 + x^4y + x + 2y$

Then, $\partial z / \partial x$ or $\partial f / \partial x$ or $f_x = 4x^3 + 3y^2 + 2xy + 1$ (here, y is consider as constant) $\partial z / \partial y$ or $\partial f / \partial y$ or $f_y = 4y^3 + 6xy + x^2 + 2$ (here, x is consider as constant)

Higher Partial Derivatives

Let f(x, y) be a function of two variables such that $\partial f / \partial x$, $\partial f / \partial y$ both exist.

(i) The partial derivative of $\partial f / \partial y$ w.r.t. 'x' is denoted by $\partial^2 f / \partial x^2$ / or fxx.

(ii) The partial derivative of $\partial f / \partial y$ w.r.t. 'y' is denoted by $\partial^2 f / \partial y^2$ / or fyy.

(iii) The partial derivative of $\partial f / \partial x$ w.r.t. 'y' is denoted by $\partial^2 f / \partial y \, \partial x$ / or fxy.

(iv) The partial derivative of $\partial f / \partial x$ w.r.t. 'x' is denoted by $\partial^2 f / \partial y \, \partial x$ / or fyx.

Note $\partial^2 f / \partial x \, \partial y = \partial^2 f / \partial y \, \partial x$

These four are second order partial derivatives.

Euler's Theorem on Homogeneous Function

If f(x, y) be a homogeneous function in x, y of degree n, then x (&partf / ∂x) + y (&partf / ∂y) = nf

Deduction Form of Euler's Theorem

If f(x, y) is a homogeneous function in x, y of degree n, then

(i) $x (\partial^2 f / \partial x^2) + y (\partial^2 f / \partial x \, \partial y) = (n - 1)$ &partf / ∂x

(ii) $x (\partial^2 f / \partial y \, \partial x) + y (\partial^2 f / \partial y^2) = (n - 1)$ &partf / ∂y

(iii) $x^2 (\partial^2 f / \partial x^2) + 2xy (\partial^2 f / \partial x \, \partial y) + y^2 (\partial^2 f / \partial y^2) = n(n - 1) f(x, y)$

Important Points to be Remembered

If α is m times repeated root of the equation f(x) = 0, then f(x) can be written as $f(x) = (x - \alpha)^m g(x)$, where $g(\alpha) \ne 0$.

From the above equation, we can see that f(α) = 0, f' (α) = 0, f'' (α) = 0, … , $f^{(m-1)}$,(α) = 0.

Hence, we have the following proposition f(α) = 0, f' (α) = 0, f'' (α) = 0, … , $f^{(m-1)}$,(α) = 0.

Therefore, α is m times repeated root of the equation f(x) = 0.

Chapter-5

Continuity and Differentiability

- A real valued function is continuous at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.

- Sum, difference, product and quotient of continuous functions are continuous. i.e., if f and g are continuous functions, then

  $(f \pm g)\ (x)\ = f\ (x)\ \pm g(x)$ is continuous.

  $(f\ .\ g)\ (x) = f\ (x)\ .\ g(x)$ is continuous.

  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (wherever $g\ (x) \neq 0$) is continuous.

- Every differentiable function is continuous, but the converse is not true.

- Chain rule is rule to differentiate composites of functions. If f = v o u, t = u (x) and if both

  and if both $\frac{dt}{dx}$ *and* $\frac{dv}{dt}$ exist then $\frac{df}{dx} = \frac{dv}{dt} = \frac{dt}{dx}$

- Following are some of the standard derivatives (in appropriate domains):

  $$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

  $$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

  $$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

  $$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{1+x^2}$$

  $$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)=\frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(e^x\right)=e^x$$

$$\frac{d}{dx}(\log x)=\frac{1}{x}$$

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$ Here both f (x) and u (x) need to be positive for this technique to make sense.
- **Rolle's Theorem:** If f: [a, b] → R is continuous on [a, b] and differentiable on (a, b) such that f (a) = f (b), then there exists some c in (a, b) such that f ′(c) = 0.
- Mean Value Theorem: If f: [a, b] → R is continuous on [a, b] and differentiable on (a, b). Then there exists some c in (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$