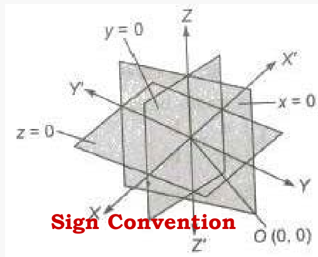
Chapter 11: Three Dimensional Geometry

Coordinate System

The three mutually perpendicular lines in a space which divides the space into eight parts and if these perpendicular lines are the coordinate axes, then it is said to be a coordinate system.



Sign Convention

Octant Coordinate	x	y	z
OXYZ	+	+	+
OX'YZ	−	+	+
OXY'Z	+	−	+
OXYZ'	+	+	−
OX'Y'Z	−	−	+
OX'YZ'	−	+	−
OXY'Z'	+	−	−
OX'Y'Z'	−	−	−

Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. The distance between these points is given by

$PQ \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

The distance of a point $P(x, y, z)$ from origin O is $OP = \sqrt{x^2 + y^2 + z^2}$

Section Formulae

(i) The coordinates of any point, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio m : n internally are $(mx_2 + nx_1 / m + n, my_2 + ny_1 / m + n, mz_2 + nz_1 / m + n)$

(ii) The coordinates of any point, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio m : n externally are $(mx_2 - nx_1 / m - n, my_2 - ny_1 / m - n, mz_2 - nz_1 / m - n)$

(iii) The coordinates of mid-point of P and Q are $(x_1 + x_2 / 2 , y_1 + y_2 / 2, z_1 + z_2 / 2)$

(iv) Coordinates of the centroid of a triangle formed with vertices P(x , y , z) and Q(x , y , z) and

1 1 1 2 2 2

$R(x_3, y_3, z_3)$ are $(x_1 + x_2 + x_3 / 3, y_1 + y_2 + y_3 / 3, z_1 + z_2 + z_3 / 3)$

(v) Centroid of a Tetrahedron

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then its centroid G is given by $(x_1 + x_2 + x_3 + x_4 / 4, y_1 + y_2 + y_3 + y_4 / 4, z_1 + z_2 + z_3 + z_4 / 4)$

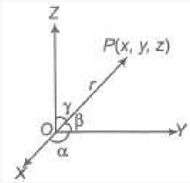
Direction Cosines

If a directed line segment OP makes angle α, β and γ with OX , OY and OZ respectively, then Cos α, cos β and cos γ are called direction cosines of up and it is represented by l, m, n.

i.e.,

l = cos α

m = cos β and n = cos γ



If OP = r, then coordinates of OP are (lr, mr , nr)

(i) If 1, m, n are direction cosines of a vector r, then

(a) r = |r| (li + mj + nk)   r = li + mj + nk

(b) $1^2 + m^2 + n^2 = 1$

(c) Projections of r on the coordinate axes are

(d) |r| = 1|r|, m|r|, n|r| / √sum of the squares of projections of r on the coordinate axes

(ii) If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, such that the direction cosines of PQ are l, m, n. Then, $x_2 - x_1 = 1|PQ|$, $y_2 - y_1 = m|PQ|$, $z_2 - z_1 = n|PQ|$ These are projections of PQ on X , Y and Z axes, respectively.

(iii) If 1, m, n are direction cosines of a vector r and a b, c are three numbers, such that l / a = m / b = n / c.

Then, we say that the direction ratio of r are proportional to a, b, Also, we have $l = a / \sqrt{a_2 + b_2 + c_2}$, $m = b / \sqrt{a_2 + b_2 + c_2}$, $n = c / \sqrt{a_2 + b_2 + c_2}$

(iv) If θ is the angle between two lines having direction cosines $1_1, m_1, n_1$ and $1_2, m_2, n_2$, then cos θ $= 1_1 1_2 + m_1 m_2 + n_1 n_2$

(a) Lines are parallel, if $l_1 / 1_2 = m_1 / m_2 = n_1 / n_2$

(b) Lines are perpendicular, if $l_1 l_2 + m_1 m_2 + n_1 n_2$

(v) If θ is the angle between two lines whose direction ratios are proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then the angle θ between them is given by cos θ $= a_1 a_2 + b_1 b_2 + c_1 c_2 / \sqrt{a^2{}_1 + b^2{}_1 + c^2{}_1} \sqrt{a^2{}_2 + b^2{}_2 + c^2{}_2}$

Lines are parallel, if $a_1 / a_2 = b_1 / b2 = c_1 / c2$

Lines are perpendicular, if $a_1 a2 + b_1 b_2 + c_1 c_2 = 0$.

(vi) The projection of the line segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ to the line having direction cosines 1, m, n is $|(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n|$.

(vii) The direction ratio of the line passing through points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are proportional to $x_2 - x_1, y_2 - y_1 - z_2 - z_1$

Then, direction cosines of PQ are$x_2 - x_1 / |PQ|$, $y_2 - y_1 / |PQ|$, $z_2 - z_1 / |PQ|$

Area of Triangle

If the vertices of a triangle be $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then

$$\text{Area of } \Delta ABC = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$$\text{where, } \Delta x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \Delta y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} \text{ and } \Delta z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Angle Between Two Intersecting Lines

If $l(x_1, m_1, n_1)$ and $l(x_2, m_2, n_2)$ be the direction cosines of two given lines, then the angle θ between them is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(i) The angle between any two diagonals of a cube is $\cos^{-1}(1 / 3)$.

(ii) The angle between a diagonal of a cube and the diagonal of a face (of the cube is $\cos^{-1}(\sqrt{2} / 3)$

Straight Line in Space

The two equations of the line $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ together represents a straight line.

1. Equation of a straight line passing through a fixed point $A(x_1, y_1, z_1)$ and having direction ratios a, b, c is given by $x - x_1 / a = y - y_1 / b = z - z_1 / c$, it is also called the symmetrically form of a line.

Any point P on this line may be taken as $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$, where λ R is parameter. If a, b, c are replaced by direction cosines 1, m, n, then λ, represents distance of the point P from the fixed point A.

2. Equation of a straight line joining two fixed points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $x - x_1 / x_2 - x_1 = y - y_1 / y_2 - y_1 = z - z_1 / z_2 - z_1$

3. Vector equation of a line passing through a point with position vector a and parallel to vector b is r = a + λ b, where A, is a parameter.

4. Vector equation of a line passing through two given points having position vectors a and b is r = a + λ (b – a) , where λ is a parameter.

5. (a) The length of the perpendicular from a point $P(\vec{\alpha})$ on the line r – a + λ b is given by

$$\sqrt{|\vec{\alpha} - \mathbf{a}|^2 - \left\{\frac{(\vec{\alpha} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|}\right\}^2}$$

(b) The length of the perpendicular from a point $P(x_1, y_1, z_1)$ on the line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \text{ is given by}$$

$$\sqrt{\{(a-x_1)^2 + (b-y_1)^2 + (c-z_1)^2\} - \{(a-x_1)\,l + (b-y_1)\,m + (c-z_1)\,n\}^2}$$

where, 1, m, n are direction cosines of the line.

6. Skew Lines Two straight lines in space are said to be skew lines, if they are neither parallel nor intersecting.

7. Shortest Distance If l_1 and l_2 are two skew lines, then a line perpendicular to each of lines 4 and 12 is known as the line of shortest distance.

If the line of shortest distance intersects the lines l_1 and l_2 at P and Q respectively, then the distance PQ between points P and Q is known as the shortest distance between l_1 and l_2.

8. The shortest distance between the lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

and

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \text{ is given by}$$

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}}$$

9. The shortest distance between lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ is given by

$$d = \left|\frac{(\mathbf{b}_1 \times \mathbf{b}_2) \cdot (\mathbf{a}_2 - \mathbf{a}_1)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$$

10. The shortest distance parallel lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ is given by

$$d = \left|\frac{(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}}{|\mathbf{b}|}\right|$$

11. Lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ are intersecting lines, if $(b_1 * b_2) * (a_2 - a_1) = 0$.

12. The image or reflection (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by $x - x_1 / a = y - y_1 / b = z - z_1 / c = -2(ax_1 + by_1 + cz1 + d) / a_2 + b_2 + c_2$

13. The foot (x, y, z) of a point (x_1, y_1, z_1) in a plane ax + by + cz + d = 0 is given by $x - x_1 / a = y - y_1 / b = z - z_1 / c = -(ax_1 + by_1 + cz_1 + d) / a_2 + b_2 + c_2$

14. Since, x, y and z-axes pass through the origin and have direction cosines (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively. Therefore, their equations are

x – axis : x – 0 / 1 = y – 0 / 0 = z – 0 / 0

y – axis : x – 0 / 0 = y – 0 / 1 = z – 0 / 0z – axis : x – 0 / 0 = y – 0 / 0 = z – 0 / 1

Plane

A plane is a surface such that, if two points are taken on it, a straight line joining them lies wholly in the surface.

General Equation of the Plane

The general equation of the first degree in x, y, z always represents a plane. Hence, the general equation of the plane is ax + by + cz + d = 0. The coefficient of x, y and z in the cartesian equation of a plane are the direction ratios of normal to the plane.

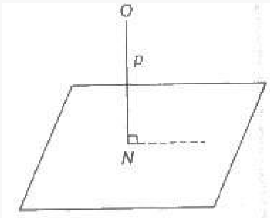
Equation of the Plane Passing Through a Fixed Point

The equation of a plane passing through a given point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Normal Form of the Equation of Plane

(i) The equation of a plane, which is at a distance p from origin and the direction cosines of the normal from the origin to the plane are l, m, n is given by lx + my + nz = p.

(ii) The coordinates of foot of perpendicular N from the origin on the plane are (1p, mp, np).



Intercept Form

The intercept form of equation of plane represented in the form of x / a + y / b + z / c = 1 where, a, b and c are intercepts on X, Y and Z-axes, respectively.

For x intercept Put y = 0, z = 0 in the equation of the plane and obtain the value of x. Similarly, we can determine for other intercepts.

Equation of Planes with Given Conditions

(i) Equation of a plane passing through the point $A(x_1, y_1, z_1)$ and parallel to two given lines with direction ratios

$$a_1, b_1, c_1 \text{ and } a_2, b_2, c_2 \text{ is } \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

(ii) Equation of a plane through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ and parallel to a line with direction ratios a, b, c is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0.$$

(iii) The Equation of a plane passing through three points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

(iv) Four points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0.$$

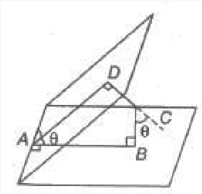
(v) Equation of the plane containing two coplanar lines and $$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

$$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \text{ is}$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Angle between Two Planes

The angle between two planes is defined as the angle between the normal to them from any point. Thus, the angle between the two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$



is equal to the angle between the normals with direction cosines $\pm a_1 / \sqrt{\Sigma a^2{}_1}, \pm b_1 / \sqrt{\Sigma a^2{}_1}, \pm c_1 / \sqrt{\Sigma a^2{}_1}$ and $\pm a_2 / \sqrt{\Sigma a^2{}_2}, \pm b_2 / \sqrt{\Sigma a^2{}_2}, \pm c2 / \sqrt{\Sigma a^2{}_2}$

If θ is the angle between the normals, then $\cos\theta = \pm a_1a_2 + b_1b_2 + c_1c_2 / \sqrt{a^2{}_1 + b^2{}_1 + c^2{}_1}\sqrt{a2{}_2 + b2{}_2 + c2{}_2}$

Parallelism and Perpendicularity of Two Planes

Two planes are parallel or perpendicular according as the normals to them are parallel or perpendicular.

Hence, the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel, if $a_1 / a_2 = b_1 / b_2 = c_1 / c_2$ and perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Note The equation of plane parallel to a given plane $ax + by + cz + d = 0$ is given by $ax + by + cz + k = 0$, where k may be determined from given conditions.

Angle between a Line and a Plane

In Vector Form The angle between a line $r = a + \lambda b$ and plane $r *\bullet n = d$, is defined as the complement of the angle between the line and normal to the plane:

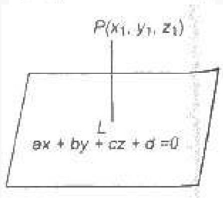
$\sin\theta = n * b / |n||b|$

In Cartesian Form The angle between a line $x - x_1 / a_1 = y - y_1 / b_1 = z - z_1 / c_1$ and plane $a_2x + b_2y + c_2z + d_2 = 0$ is $\sin\theta = a_1a_2 + b_1b_2 + c_1c / \sqrt{a^2{}_1 + b^2{}_1 + c^2{}_1}\sqrt{a^2{}_2 + b^2{}_2 + c^2{}_2}$

Distance of a Point from a Plane

Let the plane in the general form be $ax + by + cz + d = 0$. The distance of the point $P(x_1, y_1, z_1)$ from the plane is equal to

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$



If the plane is given in, normal form $lx + my + nz = p$. Then, the distance of the point $P(x_1, y_1, z_1)$ from the plane is $|lx_1 + my_1 + nz_1 - p|$.

Distance between Two Parallel Planes

If $ax + by + cz + d1 = 0$ and $ax + by + cz + d_2 = 0$ be equation of two parallel planes. Then, the distance between them is

$$\left|\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Bisectors of Angles between Two Planes

The bisector planes of the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 / \sqrt{\Sigma a^2}_1 = \pm a_2x + b_2y + c_2z + d_2 / \sqrt{\Sigma a^2}_2$

One of these planes will bisect the acute angle and the other obtuse angle between the given plane.

Sphere

A sphere is the locus of a point which moves in a space in such a way that its distance from a fixed point always remains constant.

General Equation of the Sphere

In Cartesian Form

The equation of the sphere with centre (a, b, c) and radius r is $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ …….(i)

In generally, we can write $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Here, its centre is $(-u, v, w)$ and radius $= \sqrt{u^2 + v^2 + w^2 - d}$

In Vector Form

The vector equation of a sphere of radius a and Centre having position vector c is $|r - c| = a$

Important Points to be Remembered

(i) The general equation of second degree in x, y, z is $ax^2 + by^2 + cz^2 + 2hxy + 2kyz + 2lzx + 2ux + 2vy + 2wz + d = 0$ represents a sphere, if

(a) $a = b = c\ (\neq 0)$

(b) $h = k = 1 = 0$

The equation becomes $ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d - 0$ …(A)

To find its centre and radius first we make the coefficients of x^2, y^2 and z^2 each unity by dividing throughout by a.

Thus, we have $x^2+y^2+z^2 + (2u / a)\,x + (2v / a)\,y + (2w / a)\,z + d / a = 0$ …..(B)

Centre is $(-u / a, -v / a, -w / a)$

and radius $= \sqrt{u^2 / a^2 + v^2 / a^2 + w^2 / a^2 - d / a}$

$= \sqrt{u^2 + v^2 + w^2 - ad} / |a|$.

(ii) Any sphere concentric with the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + k = 0$

(iii) Since, $r^2 = u^2 + v^2 + w^2 - d$, therefore, the Eq. (B) represents a real sphere, if $u^2 + v^2 + w^2 - d > 0$

(iv) The equation of a sphere on the line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) as a diameter is

$(x - x_1)(x - x_1) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$

(v) The equation of a sphere passing through four non-coplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

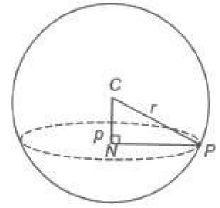
Tangency of a Plane to a Sphere

The plane $lx + my + nz = p$ will touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if length of the perpendicular from the centre $(-u, -v, -w)$ = radius, i.e., $|lu - mv - nw - p| / \sqrt{l^2 + m^2 + n^2}$ $= \sqrt{u^2 + v^2 + w^2 - d}$ $(lu - mv - nw - p)^2 = (u^2 + v^2 + w^2 - d)(l^2 + m^2 + n^2)$

Plane Section of a Sphere

Consider a sphere intersected by a plane. The set of points common to both sphere and plane is called a plane section of a sphere. In ΔCNP, $NP^2 = CP^2 - CN^2 = r^2 - p^2$

$NP = \sqrt{r^2 - p^2}$



Hence, the locus of P is a circle whose centre is at the point N, the foot of the perpendicular from the centre of the sphere to the plane.

The section of sphere by a plane through its centre is called a great circle. The centre and radius of a great circle are the same as those of the sphere.

Chapter-11

Three Dimensional Geometry

Direction cosines of a line are the cosines of the angles made by the line with the positive direct ions of the coordinate axes.

- *If* l, m, n are the direct ion cosines of a line, then $l^2 + m^2 + n^2 = 1$
- Direct ion cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_2}{PQ}$
- Where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- Direction ratios of a line are the numbers which are proportional to the direct ion cosines of a line.
- If l, m, n are the direct ion cosines and a, b, c are the direct ion ratios of a line

Then, $l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$

- Skew lines are lines in space which are neither parallel nor intersecting. They lie in different planes.
- Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then,

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Vector equation of a line that passes through the given point whose position vector is $\bar{a}$ and parallel to a given vector $\bar{b}$ is $\bar{r} = \bar{a} + \lambda \bar{b}$

- Equation of a line through a point (x_1, y_1, z_1) and having direct ion cosines $l,\ m,\ n$ is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

- The vector equation of a line which passes through two points whose posit ion vectors are $\bar{a}$ and $\bar{b}$ is $\bar{r}=\bar{a}+\lambda(\bar{b}-\bar{a})$

- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

- If θ is the acute angle between $\bar{r}=\bar{a}_1+\lambda\bar{b}_1$ and $\bar{r}=\bar{a}_2+\lambda\bar{b}_2$ then, $\cos\theta=\left|\frac{\bar{b}.\bar{b}_2}{|\bar{b}_1||\bar{b}_2|}\right|$

- If $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta=|l_1l_2+m_1m_2+n_1n_2|$

- Shortest distance between two skew lines is the line segment perpendicular to both the lines.

- Shortest distance between $\bar{r}=\bar{a}_1+\wedge\bar{b}_1$ and $\bar{r}=\bar{a}_2+\wedge\bar{b}_2$ $\left|\frac{(\bar{b}_1\times\bar{b}_2).(\bar{a}_2-\bar{a}_1)}{|\bar{b}_1\times\bar{b}_2|}\right|$

- Shortest distance between the lines: $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_1}=\frac{y-y_2}{b_1}=\frac{z-z_2}{c_1}$ is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

- Distance between parallel lines $\bar{r}=\bar{a}_1+\wedge\bar{b}_1$ and $\bar{r}=\bar{a}_2+\wedge\bar{b}_2$ $\left|\frac{(\bar{b})\times(\bar{a}_2-\bar{a}_1)}{|\bar{b}_1|}\right|$

- In the vector form, equation of a plane which is at a distance d from the origin, and n ^ is the unit vector normal to the plane through the origin is $\bar{r}.\hat{n} = d$
- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose posit ion vector is a and perpendicular to the vector $\bar{N}$ $is(\bar{r} - \bar{a})$. N=0
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is
- $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$
- Equation of a plane passing through three non collinear points (x_1, y_1, z_1)
- $(x_2, y_2, z_2)$and (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
- Vector equation of a plane that contains three non collinear points having position vectors
- $\bar{a}, \bar{b}$ and $\bar{c}$ is $(\bar{r} - \bar{a}).\left[(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})\right] = 0$
- Equation of a plane that cuts the coordinates axes at $(a,\ 0,\ 0),\ (0,\ b,\ 0)$ *and* $(0,\ 0,\ c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
- Vector equation of a plane that p asses thro ugh the in the section of planes $\bar{r}.\bar{n}_1 = d_1$ and $\bar{r}.\bar{n}_2 = d_2$ is $\bar{r}.(\bar{n}_1 + \lambda\bar{n}_2) = d_1 + \lambda d_2$ where λ is any nonzero constant.
- Cartesian equation of a plane that passes that passes through the intersection of two given planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is $(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2 = 0$
- Two lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}_1$and $\bar{r} = \bar{a}_2 + \mu\bar{b}_2$ are coplanar if $(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times -\bar{b}_2) = 0$

- In the Cartesian form above lines passing through the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2 z_2)$

- $= \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{C_2}$ are coplanar if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- In the vector form, if θ is the angle between the two planes, $\overline{r}.\overline{n}_1 = d_1$ and $\overline{r}.\overline{n}_2 = d_2$, then $\theta = \cos^{-1}\dfrac{|\overline{n}_1.\overline{n}_2|}{|\overline{n}_1||\overline{n}_2|}$

- The angle ϕ between the line $\overline{r} = \overline{a} + \lambda\overline{b}$ and the plane $\overline{r}.\hat{n} = d$ $\sin\phi = \left|\dfrac{\overline{b}.\hat{n}}{|\overline{b}|\,|\hat{n}|}\right|$

- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by $\cos\theta = \left|\dfrac{A_1A_2 + +B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\,\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$

- The distance of a point whose position vector is $\overline{a}$ from the plane $\overline{r}.\hat{n} = d$ is $|d - \overline{a}.\hat{n}|$

- The distance from a point (x_1, y_1, z_1) to the plane Ax + By + Cz + D = 0 is $\left|\dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$