Relations and Functions

1. What is Cartesian Sets?

Given two non-empty sets A and B. The Cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B, i.e.,

$A \times B = \left\{ {\left({a,b} \right):a \in A,b \in B} \right\}$

If either A or B is the null set, then $A \times B$ will also be empty set, i.e., $A \times B$

Important tips

Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal.

If there are p elements in A and q elements in B, then there will be pq elements in $A \times B$, i.e., if $n\left(A \right) = p$ and $n\left(B \right) = q$, then $n\left({A \times B} \right) = pq$.

If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$.

$A \times A \times A = \left\{ {\left({a,b,c} \right):a,b,c \in A} \right\}$. Here ${\left({a,b,c} \right)}$ is called an *ordered triplet*

Questtion 1 If $P = \left\{ {1,2} \right\}$ and $Q = \left\{ {5,4,2} \right\}$, find $P \times Q$ and $Q \times P$.

Solution

$P = \left\{ {1,2} \right\}$ and $Q = \left\{ {5,4,2} \right\}$

We know that the Cartesian product $P \times Q$ of two non-empty sets P and Q is defined as

$P \times Q = \left\{ {\left({p,q} \right):p \in P,q \in Q} \right\}$

$\therefore P \times Q = \left\{ {\left({1,5} \right),\left({1,4} \right),\left({1,2} \right),\left({2,5} \right),\left({2,4} \right),\left({2,2} \right)} \right\}$

$Q \times P = \left\{ {\left({5,1} \right),\left({5,2} \right),\left({4,1} \right),\left({4,2} \right),\left({2,1} \right),\left({2,2} \right)} \right\}$

Question 2 If $A = \left\{ { - 1,1} \right\}$, find $A \times A \times A$

Solution

It is known that for any non-empty set A, $A \times A \times A$ is defined as

$A \times A \times A = \left\{ {\left({a,b,c} \right):a,b,c \in A} \right\}$

It is given that $A = \left\{ { - 1,1} \right\}$

$\therefore A \times A \times A\left\{ {\left({ - 1, - 1, - 1} \right),\left({ - 1, - 1,1} \right),\left({ - 1,1, - 1} \right),\left({ - 1,1,1} \right),\left({1, - 1, - 1} \right),\left({1, - 1,1} \right),\left({1,1, - 1} \right),\left({1,1,1} \right),} \right\}$

2. What is relations?

A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$.

It *"maps"* elements of one set to another set. The subset is derived by describing a relationship between the first element and the second element of the ordered pair $\left({A \times B} \right)$.

Domain: The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

Range: the set of all the ending points is called the *range*

A relation can be expressed in Set builder or Roaster form

Roster forms

In a Roster forms, all the elements in the set is listed.

Example

Set of $vovel = \left\{ {a,e,i,o,u} \right\}$

Some Important points

In roster form, the order in which the elements are listed is immaterial

while writing the set in roster form an element is not generally repeated

Set Builder Form

In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set. For example, in the set $\left\{ {2,4,6,8} \right\}$, all the elements possess a common property, namely, each of them is a even number less than 10. Denoting this set by N, we write

N = {x : x is a even number less than 10 }

b) We describe the element of the set by using a symbol x (any other symbol like the letters y, z, etc. could be used) which is followed by a colon " : ". After the sign of colon, we write the characteristic property possessed by the elements of the set and then enclose the whole description within braces

Important Note

The total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \cdot B$. If $n\left(A \right) = p$ and $n\left(B \right) = q$, then $n\left({A \cdot B} \right) = pq$ and the total number of relations is ${2^{pq}}$

Example:

Let $P = \left\{ {1,2,3,.....,18} \right\}$ define a relation R from P to P by $R = \left\{ {\left({x,y} \right):2x - y = 0,where{\rm{ }}x,y \in P} \right\}$ Write down its domain, codomain and range.

Solution: The relation R from P to P is given as

R = {(x,y):2x-y=0, where x, y $\in$ P}

i.e., R = {(x, y): 2x = y, where x, y $\in$ P}

$\therefore R = \left\{ \left(1, 2 \right), \left(2, 4 \right), \left(3, 6 \right), \left(4, 8 \right), \left(5, 10 \right), \left(6, 12 \right), \left(7, 14 \right), \left(8, 16 \right), \left(9, 18 \right) \right\}$

The domain of R is the set of all first elements of the ordered pairs in the relation.

$\therefore \text{Domain of } R = \left\{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \right\}$

The whole set P is the codomain of the relation R.

Therefore codomain of $R = P = \left\{ 1, 2, 3, \ldots, 18 \right\}$

The range of R is the set of all second elements of the ordered pairs in the relation.

Therefore range of $R = \left\{ 2, 4, 6, 8, 10, 12, 14, 16, 18 \right\}$

3. What is Function

A function is a "well-behaved" relation

A function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

For a relation to be a function, there must be only and exactly one y that corresponds to a given x

If f is a function from A to B and $\left(a, b \right) \in f$, then $f\left(a \right) = b$, where b is called the image of a under f and a is called the preimage of b under f.

Example 1:

Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

$\left\{ \left(3,1 \right), \left(5,1 \right), \left(7,1 \right), \left(11,1 \right), \left(14,1 \right), \left(17,1 \right) \right\}$

$\left\{ \left(2,1 \right), \left(4,2 \right), \left(6,3 \right), \left(6,4 \right), \left(10,5 \right), \left(12,6 \right), \left(14,7 \right) \right\}$

$\left\{ \left(1,3 \right), \left(1,5 \right), \left(2,5 \right) \right\}$

Answer

$\left\{ \left(3,1 \right), \left(5,1 \right), \left(7,1 \right), \left(11,1 \right), \left(14,1 \right), \left(17,1 \right) \right\}$

Since 3, 5, 8, 11, 14, and 17 are the elements of the domain of the given relation having their unique images, this relation is a function.

$\left\{ \left(2,1 \right), \left(4,2 \right), \left(6,3 \right), \left(6,4 \right), \left(10,5 \right), \left(12,6 \right), \left(14,7 \right) \right\}$

Since the same first element i.e 6 corresponds to two different images 3 and 4, this relation is not a function

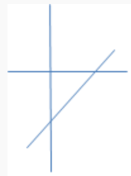
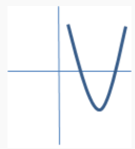
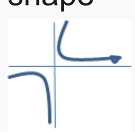
$\left\{ \left(1,3 \right), \left(1,5 \right), \left(2,5 \right) \right\}$

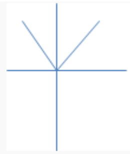
Since the same first element i.e., 1 corresponds to two different images i.e., 3 and 5, this relation is not a function.

Important functions

Let us take some useful polynomial and shapes obtained on the Cartesian plane

S.No.	$y = p\left(x \right)$	Graph	Name of the graph	Name of the

		obtained		function
1.	$y = mx + c$ where m and c can be any values $\left({m \ne 0} \right)$ Example $y = 2x + 3$		Graphs of these functions are straight lines. m is the slope and b is the y intercept. If m is positive then the line rises to the right and if m is negative then the line falls to the right	Linear function. Typical use for linear functions is converting from one quantity or set of units to another.
2.	$y = a{x^2} + bx + c$ where, ${b^2} - 4ac{\rm{ }} > {\rm{ }}0$, $a \ne 0$ and $a > {\rm{ }}0$ example- $y = {x^2} - 7x + 12$		Parabola It intersect the x- axis at two points Example- (3,0) and (4,0)	Quadratic function
3.	$y = a{x^3} + b{x^2} + cx + d$ where,$a \ne 0$	It can be of any shape 	It will cut the x-axis at the most 3 times	Cubic Function
4.	${a_n}{x^n} + {a_{n - 1}}{x^{n - 1}} + {a_{n - 2}}{x^{n - 2}} + \ldots \ldots \ldots \ldots + ax + {a_0}$ where ${a_n} \ne 0$	It can be of any shape 	It will cut the x-axis at the most n times	Polynomial function
5.	$y = \frac{{f(x)}}{{g(x)}}$ where $g(x) \ne 0$ example- $y = \frac{1}{x}$	It can be any shape 	An asymptote is a line that the curve approaches but does not cross.There are vertical and horizontal asymptote	Rational function
6.	$y = \left\| x \right\|$ i.e., $y = - x$ for $x < 0$			Modulus

	$y = x$ for $x \ge 0$			function
7.	$y = a\ln\left(x \right) + b$ where x is in the natural logarithm and a and b are constants They are only defined for positive		For small x they are negative and for large x they are positive x	Logarithmic functions
8.	$y = \left[x \right]$ $\left[x \right]{\rm{ }} - > $the value of the greatest integer, less than or equal to x			Greatest integer function

4. Algebra of Real Function

Real Value Function: A function which has all real number or subset of the real number as it domain

Real Valued Function: A function which has all real number or subset of the real number as it range

For functions $f:{\rm{ }}X - > {\bf{R}}$ and $g:{\rm{ }}X - > {\bf{R}}$, we have

Addition

$\left({f + g} \right)\left(x \right) = f\left(x \right) + g\left(x \right),x \in X$

Substraction

$\left({f - g} \right)\left(x \right) = f\left(x \right)-g\left(x \right),x \in X$

Multiplication

$\left({f.g} \right)\left(x \right) = f\left(x \right).g\left(x \right),x \in X$

Multiplication by real number $\left({kf} \right)\left(x \right) = k{\rm{ }}f\left(x \right),x \in X$, where k is a real number.

Division

$\frac{f}{g}\left(x \right) = \frac{{f(x)}}{{g(x)}}$

$x \in X$ and $g\left(x \right) \ne 0$

Chapter 2:
Relations and Functions

Concept:

Cartesian products of sets – equality of ordered pairs- triple product- relations- functions- domain- range- different types of functions- algebra of functions.

Notes:

- If (a,b) = (c,d) then a = c and b = d.
- AxB = { (x,y) / x ϵ A, y ϵ B }
- AxAxA = { (x,y,z) / x, y, z ϵ A }
- A relation R is a subset of the Cartesian product.
- A function is a relation with every element of first set has one only one image in second set.
- The set of all first elements of the ordered pairs in a function is called domain.
- The set of all second elements of the ordered pairs in a function is called the range.
- Second set itself is known as co-domain.

Text book questions

Ex: 2.1	Questions: 1, 2^*, 5^*, 7^*
Ex: 2.2	Questions: 1, 2, 6, 7^*
Ex:2.3	Questions: 2^*, 5^*
Misc. Ex:	Questions: 3^*, 4, 6, 8, 11, 12
Example	Question: 22^*

Extra/HOT questions

1. Find x and y if $(x^2-3x, y^2-5y) = (-2, -6)$.
2. Draw he graph of the following functions:
   a) Modulus function in [-4, 4]
   b) Signum function in [-6, 6]
   c) Greatest integer function in [-3, 4]

3. Find the domain of the following functions:
   a) $f(x) = \frac{x^2-1}{x-1}$
   b) $f(x) = \frac{3x+1}{x^2-5x+6}$
   c) $f(x) = \frac{2x-3}{(x-1)(x+2)}$

4. Find the domain and range of the following functions:
   a) $f(x) = \frac{1}{9-x^2}$
   b) $f(x) = \sqrt{x^2 - 1}$
   c) $f(x) = \frac{1}{x^2+4}$
   d) $f(x) = \frac{|x|}{1+|x|}$
5. If $f(x) = x^2 + \frac{1}{x^2}$ then show that f(a) = f(1/a) and also evaluate f(3/2)-f(2/3)
6. Let R = {(x,y) / x, y ϵ N , x+2y = 13} then write R as an ordered pair and also find the domain and range.
7. Let A = {x / x is a natural number <12 } and R be a relation in A defined by (x,y) in R if x+y = 12, then write R.
8. A function f is defined on the set of natural numbers as

$$f(x) = \begin{cases} x^2 \; if \; 1 \leq x < 5 \\ x + 3 \; if \; 5 < x \leq 8 \\ \dfrac{x-3}{2} \; if \; 8 < x \leq 11 \end{cases}$$

Write the function in roster form and also find the domain and range of the function.

9. Let A = {1,2,3,4}, B = {-1, 0, 1} and C = {3, 4} then verify the following:
   a) A X (B U C) = (A X B) U (A X C)
   b) A X (B-C) = (A X B) – (A X C)
   c) $A \; X \; (B \cap C) = (A \; X \; B) \cap (A \; X \; C)$
10. If A = {-3, -2, 0, 2, 3} write the subset B of A X A such that first element of B is either -3 or +3.