

NCERT Solutions for Class 11th Maths: Chapter 3 – Trigonometric Functions

Exercise 3.1

Question 1:

Find the radian measures corresponding to the following degree measures:

(i) 25° (ii) $-47^\circ 30'$ (iii) 240° (iv) 520°

Answer

(i) 25°

We know that $180^\circ = \pi$ radian

$$\therefore 25^\circ = \frac{\pi}{180} \times 25 \text{ radian} = \frac{5\pi}{36} \text{ radian}$$

(ii) $-47^\circ 30'$

$$-47^\circ 30' = -47\frac{1}{2} \text{ degree } [1^\circ = 60']$$

$$= -\frac{95}{2} \text{ degree}$$

Since $180^\circ = \pi$ radian

$$\frac{-95}{2} \text{ deg ree} = \frac{\pi}{180} \times \left(\frac{-95}{2}\right) \text{ radian} = \left(\frac{-19}{36 \times 2}\right) \pi \text{ radian} = \frac{-19}{72} \pi \text{ radian}$$

$$\therefore -47^\circ 30' = \frac{-19}{72} \pi \text{ radian}$$

(iii) 240°

We know that $180^\circ = \pi$ radian

$$\therefore 240^\circ = \frac{\pi}{180} \times 240 \text{ radian} = \frac{4}{3} \pi \text{ radian}$$

(iv) 520°

We know that $180^\circ = \pi$ radian

$$\therefore 520^\circ = \frac{\pi}{180} \times 520 \text{ radian} = \frac{26\pi}{9} \text{ radian}$$

Question 2:

Find the degree measures corresponding to the following radian measures

$$\left(\text{Use } \pi = \frac{22}{7} \right)$$

$$(i) \frac{11}{16} \text{ (ii) } -4 \text{ (iii) } \frac{5\pi}{3} \text{ (iv) } \frac{7\pi}{6}$$

Answer

$$(i) \frac{11}{16}$$

We know that n radian = 180°

$$\begin{aligned}\therefore \frac{11}{16} \text{ radian} &= \frac{180}{\pi} \times \frac{11}{16} \text{ deg ree} = \frac{45 \times 11}{\pi \times 4} \text{ deg ree} \\ &= \frac{45 \times 11 \times 7}{22 \times 4} \text{ deg ree} = \frac{315}{8} \text{ deg ree} \\ &= 39 \frac{3}{8} \text{ deg ree} \\ &= 39^\circ + \frac{3 \times 60}{8} \text{ min utes} \quad [1^\circ = 60'] \\ &= 39^\circ + 22' + \frac{1}{2} \text{ min utes} \\ &= 39^\circ 22' 30'' \quad [1' = 60'']\end{aligned}$$

(ii) - 4

We know that n radian = 180°

$$\begin{aligned}-4 \text{ radian} &= \frac{180}{\pi} \times (-4) \text{ deg ree} = \frac{180 \times 7(-4)}{22} \text{ deg ree} \\ &= \frac{-2520}{11} \text{ deg ree} = -229 \frac{1}{11} \text{ deg ree} \\ &= -229^\circ + \frac{1 \times 60}{11} \text{ min utes} \quad [1^\circ = 60'] \\ &= -229^\circ + 5' + \frac{5}{11} \text{ min utes} \\ &= -229^\circ 5' 27'' \quad [1' = 60'']\end{aligned}$$

$$(iii) \frac{5\pi}{3}$$

We know that n radian = 180°

$$\therefore \frac{5\pi}{3} \text{ radian} = \frac{180}{\pi} \times \frac{5\pi}{3} \text{ deg ree} = 300^\circ$$

$$(iv) \frac{7\pi}{6}$$

We know that n radian = 180°

$$\therefore \frac{7\pi}{6} \text{ radian} = \frac{180}{\pi} \times \frac{7\pi}{6} = 210^\circ$$

Question 3:

A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

Answer

Number of revolutions made by the wheel in 1 minute = 360

$$\therefore \text{Number of revolutions made by the wheel in 1 second} = \frac{360}{60} = 6$$

In one complete revolution, the wheel turns an angle of 2π radian.

Hence, in 6 complete revolutions, it will turn an angle of $6 \times 2\pi$ radian, i.e., 12π radian

Thus, in one second, the wheel turns an angle of 12π radian.

Question 4:

Find the degree measure of the angle subtended at the centre of a circle of radius 100

cm by an arc of length 22 cm $\left(\text{Use } \pi = \frac{22}{7} \right)$.

Answer

We know that in a circle of radius r unit, if an arc of length l unit subtends an angle θ radian at the centre, then

$$\theta = \frac{l}{r}$$

Therefore, for $r = 100$ cm, $l = 22$ cm, we have

$$\begin{aligned} \theta &= \frac{22}{100} \text{ radian} = \frac{180}{\pi} \times \frac{22}{100} \text{ deg ree} = \frac{180 \times 7 \times 22}{22 \times 100} \text{ deg ree} \\ &= \frac{126}{10} \text{ deg ree} = 12\frac{3}{5} \text{ deg ree} = 12^\circ 36' \quad [1^\circ = 60'] \end{aligned}$$

Thus, the required angle is $12^\circ 36'$.

Question 5:

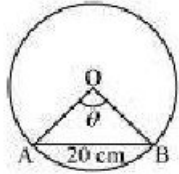
In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

Answer

Diameter of the circle = 40 cm

$$\therefore \text{Radius } (r) \text{ of the circle} = \frac{40}{2} \text{ cm} = 20 \text{ cm}$$

Let AB be a chord (length = 20 cm) of the circle.



In $\triangle OAB$, $OA = OB = \text{Radius of circle} = 20 \text{ cm}$

Also, $AB = 20 \text{ cm}$

Thus, $\triangle OAB$ is an equilateral triangle.

$$\therefore \theta = 60^\circ = \frac{\pi}{3} \text{ radian}$$

We know that in a circle of radius r unit, if an arc of length l unit subtends an angle θ

radian at the centre, then $\theta = \frac{l}{r}$.

$$\frac{\pi}{3} = \frac{\widehat{AB}}{20} \Rightarrow \widehat{AB} = \frac{20\pi}{3} \text{ cm}$$

Thus, the length of the minor arc of the chord is $\frac{20\pi}{3} \text{ cm}$.

Question 6:

If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

Answer

Let the radii of the two circles be r_1 and r_2 . Let an arc of length l subtend an angle of 60° at the centre of the circle of radius r_1 , while let an arc of length l subtend an angle of 75° at the centre of the circle of radius r_2 .

$$\text{Now, } 60^\circ = \frac{\pi}{3} \text{ radian} \quad \text{and } 75^\circ = \frac{5\pi}{12} \text{ radian}$$

We know that in a circle of radius r unit, if an arc of length l unit subtends an angle θ

$$\text{radian at the centre, then } \theta = \frac{l}{r} \text{ or } l = r\theta$$

$$\therefore l = \frac{r_1\pi}{3} \text{ and } l = \frac{r_2 5\pi}{12}$$

$$\Rightarrow \frac{r_1\pi}{3} = \frac{r_2 5\pi}{12}$$

$$\Rightarrow r_1 = \frac{r_2 5}{4}$$

$$\Rightarrow \frac{r_1}{r_2} = \frac{5}{4}$$

Thus, the ratio of the radii is 5:4.

Question 7:

Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length

(i) 10 cm (ii) 15 cm (iii) 21 cm

Answer

We know that in a circle of radius r unit, if an arc of length l unit subtends an angle θ

$$\text{radian at the centre, then } \theta = \frac{l}{r}$$

It is given that $r = 75$ cm

(i) Here, $l = 10$ cm

$$\theta = \frac{10}{75} \text{ radian} = \frac{2}{15} \text{ radian}$$

(ii) Here, $l = 15$ cm

$$\theta = \frac{15}{75} \text{ radian} = \frac{1}{5} \text{ radian}$$

(iii) Here, $l = 21$ cm

$$\theta = \frac{21}{75} \text{ radian} = \frac{7}{25} \text{ radian}$$

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Exercise 3.2

Question 1:

Find the values of other five trigonometric functions if $\cos x = -\frac{1}{2}$, x lies in third quadrant.

Answer

$$\cos x = -\frac{1}{2}$$

$$\therefore \sec x = \frac{1}{\cos x} = \frac{1}{\left(-\frac{1}{2}\right)} = -2$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \sin^2 x = 1 - \left(-\frac{1}{2}\right)^2$$

$$\Rightarrow \sin^2 x = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow \sin x = \pm \frac{\sqrt{3}}{2}$$

Since x lies in the 3rd quadrant, the value of $\sin x$ will be negative.

$$\therefore \sin x = -\frac{\sqrt{3}}{2}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)} = -\frac{2}{\sqrt{3}}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\frac{1}{2}\right)} = \sqrt{3}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\sqrt{3}}$$

Question 2:

Find the values of other five trigonometric functions if $\sin x = \frac{3}{5}$, x lies in second quadrant.

Answer

$$\sin x = \frac{3}{5}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(\frac{3}{5}\right)} = \frac{5}{3}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \cos^2 x = 1 - \sin^2 x$$

$$\Rightarrow \cos^2 x = 1 - \left(\frac{3}{5}\right)^2$$

$$\Rightarrow \cos^2 x = 1 - \frac{9}{25}$$

$$\Rightarrow \cos^2 x = \frac{16}{25}$$

$$\Rightarrow \cos x = \pm \frac{4}{5}$$

Since x lies in the 2nd quadrant, the value of $\cos x$ will be negative

$$\therefore \cos x = -\frac{4}{5}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\left(-\frac{4}{5}\right)} = -\frac{5}{4}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(\frac{3}{5}\right)}{\left(-\frac{4}{5}\right)} = -\frac{3}{4}$$

$$\cot x = \frac{1}{\tan x} = -\frac{4}{3}$$

Question 3:

Find the values of other five trigonometric functions if $\cot x = \frac{3}{4}$, x lies in third quadrant.

Answer

$$\cot x = \frac{3}{4}$$

$$\tan x = \frac{1}{\cot x} = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\Rightarrow 1 + \left(\frac{4}{3}\right)^2 = \sec^2 x$$

$$\Rightarrow 1 + \frac{16}{9} = \sec^2 x$$

$$\Rightarrow \frac{25}{9} = \sec^2 x$$

$$\Rightarrow \sec x = \pm \frac{5}{3}$$

Since x lies in the 3rd quadrant, the value of $\sec x$ will be negative.

$$\therefore \sec x = -\frac{5}{3}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\left(-\frac{5}{3}\right)} = -\frac{3}{5}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\Rightarrow \frac{4}{3} = \frac{\sin x}{\left(-\frac{3}{5}\right)}$$

$$\Rightarrow \sin x = \left(\frac{4}{3}\right) \times \left(-\frac{3}{5}\right) = -\frac{4}{5}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = -\frac{5}{4}$$

Question 4:

Find the values of other five trigonometric functions if $\sec x = \frac{13}{5}$, x lies in fourth quadrant.

Answer

$$\sec x = \frac{13}{5}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\left(\frac{13}{5}\right)} = \frac{5}{13}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \sin^2 x = 1 - \left(\frac{5}{13}\right)^2$$

$$\Rightarrow \sin^2 x = 1 - \frac{25}{169} = \frac{144}{169}$$

$$\Rightarrow \sin x = \pm \frac{12}{13}$$

Since x lies in the 4th quadrant, the value of $\sin x$ will be negative.

$$\therefore \sin x = -\frac{12}{13}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(-\frac{12}{13}\right)} = -\frac{13}{12}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\left(-\frac{12}{13}\right)}{\left(\frac{5}{13}\right)} = -\frac{12}{5}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\left(-\frac{12}{5}\right)} = -\frac{5}{12}$$

Question 5:

Find the values of other five trigonometric functions if $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Answer

$$\tan x = -\frac{5}{12}$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{\left(-\frac{5}{12}\right)} = -\frac{12}{5}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\Rightarrow 1 + \left(-\frac{5}{12}\right)^2 = \sec^2 x$$

$$\Rightarrow 1 + \frac{25}{144} = \sec^2 x$$

$$\Rightarrow \frac{169}{144} = \sec^2 x$$

$$\Rightarrow \sec x = \pm \frac{13}{12}$$

Since x lies in the 2nd quadrant, the value of $\sec x$ will be negative.

$$\therefore \sec x = -\frac{13}{12}$$

$$\cos x = \frac{1}{\sec x} = \frac{1}{\left(-\frac{13}{12}\right)} = -\frac{12}{13}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\Rightarrow -\frac{5}{12} = \frac{\sin x}{\left(-\frac{12}{13}\right)}$$

$$\Rightarrow \sin x = \left(-\frac{5}{12}\right) \times \left(-\frac{12}{13}\right) = \frac{5}{13}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\left(\frac{5}{13}\right)} = \frac{13}{5}$$

Question 6:

Find the value of the trigonometric function $\sin 765^\circ$

Answer

It is known that the values of $\sin x$ repeat after an interval of 2π or 360° .

$$\therefore \sin 765^\circ = \sin(2 \times 360^\circ + 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

Question 7:

Find the value of the trigonometric function $\operatorname{cosec}(-1410^\circ)$

Answer

It is known that the values of $\operatorname{cosec} x$ repeat after an interval of 2π or 360° .

$$\begin{aligned}\therefore \operatorname{cosec}(-1410^\circ) &= \operatorname{cosec}(-1410^\circ + 4 \times 360^\circ) \\ &= \operatorname{cosec}(-1410^\circ + 1440^\circ) \\ &= \operatorname{cosec}30^\circ = 2\end{aligned}$$

Question 8:

Find the value of the trigonometric function $\tan \frac{19\pi}{3}$

Answer

It is known that the values of $\tan x$ repeat after an interval of π or 180° .

$$\therefore \tan \frac{19\pi}{3} = \tan 6\frac{1}{3}\pi = \tan \left(6\pi + \frac{\pi}{3} \right) = \tan \frac{\pi}{3} = \tan 60^\circ = \sqrt{3}$$

Question 9:

Find the value of the trigonometric function $\sin \left(-\frac{11\pi}{3} \right)$

Answer

It is known that the values of $\sin x$ repeat after an interval of 2π or 360° .

$$\therefore \sin \left(-\frac{11\pi}{3} \right) = \sin \left(-\frac{11\pi}{3} + 2 \times 2\pi \right) = \sin \left(\frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}$$

Question 10:

Find the value of the trigonometric function $\cot \left(-\frac{15\pi}{4} \right)$

Answer

It is known that the values of $\cot x$ repeat after an interval of π or 180° .

$$\therefore \cot \left(-\frac{15\pi}{4} \right) = \cot \left(-\frac{15\pi}{4} + 4\pi \right) = \cot \frac{\pi}{4} = 1$$

Exercise 3.3

Question 1:

$$\sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} = -\frac{1}{2}$$

Answer

$$\begin{aligned} \text{L.H.S.} &= \sin^2 \frac{\pi}{6} + \cos^2 \frac{\pi}{3} - \tan^2 \frac{\pi}{4} \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - (1)^2 \\ &= \frac{1}{4} + \frac{1}{4} - 1 = -\frac{1}{2} \\ &= \text{R.H.S.} \end{aligned}$$

Question 2:

Prove that $2 \sin^2 \frac{\pi}{6} + \operatorname{cosec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} = \frac{3}{2}$

Answer

$$\begin{aligned}\text{L.H.S.} &= 2 \sin^2 \frac{\pi}{6} + \cos \operatorname{ec}^2 \frac{7\pi}{6} \cos^2 \frac{\pi}{3} \\ &= 2 \left(\frac{1}{2} \right)^2 + \cos \operatorname{ec}^2 \left(\pi + \frac{\pi}{6} \right) \left(\frac{1}{2} \right)^2 \\ &= 2 \times \frac{1}{4} + \left(-\cos \operatorname{ec} \frac{\pi}{6} \right)^2 \left(\frac{1}{4} \right) \\ &= \frac{1}{2} + (-2)^2 \left(\frac{1}{4} \right) \\ &= \frac{1}{2} + \frac{4}{4} = \frac{1}{2} + 1 = \frac{3}{2} \\ &= \text{R.H.S.}\end{aligned}$$

Question 3:

Prove that $\cot^2 \frac{\pi}{6} + \cos \operatorname{ec} \frac{5\pi}{6} + 3 \tan^2 \frac{\pi}{6} = 6$

Answer

$$\begin{aligned}\text{L.H.S.} &= \cot^2 \frac{\pi}{6} + \cos \operatorname{ec} \frac{5\pi}{6} + 3 \tan^2 \frac{\pi}{6} \\ &= (\sqrt{3})^2 + \cos \operatorname{ec} \left(\pi - \frac{\pi}{6} \right) + 3 \left(\frac{1}{\sqrt{3}} \right)^2 \\ &= 3 + \cos \operatorname{ec} \frac{\pi}{6} + 3 \times \frac{1}{3} \\ &= 3 + 2 + 1 = 6 \\ &= \text{R.H.S.}\end{aligned}$$

Question 4:

Prove that $2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3} = 10$

Answer

$$\text{L.H.S.} = 2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3}$$

$$\begin{aligned}
&= 2 \left\{ \sin \left(\pi - \frac{\pi}{4} \right) \right\}^2 + 2 \left(\frac{1}{\sqrt{2}} \right)^2 + 2(2)^2 \\
&= 2 \left\{ \sin \frac{\pi}{4} \right\}^2 + 2 \times \frac{1}{2} + 8 \\
&= 2 \left(\frac{1}{\sqrt{2}} \right)^2 + 1 + 8 \\
&= 1 + 1 + 8 \\
&= 10 \\
&= \text{R.H.S}
\end{aligned}$$

Question 5:

Find the value of:

- (i) $\sin 75^\circ$
- (ii) $\tan 15^\circ$

Answer

$$\begin{aligned}
\text{(i) } \sin 75^\circ &= \sin (45^\circ + 30^\circ) \\
&= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ
\end{aligned}$$

[$\sin (x + y) = \sin x \cos y + \cos x \sin y$]

$$\begin{aligned}
&= \left(\frac{1}{\sqrt{2}} \right) \left(\frac{\sqrt{3}}{2} \right) + \left(\frac{1}{\sqrt{2}} \right) \left(\frac{1}{2} \right) \\
&= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$

$$\text{(ii) } \tan 15^\circ = \tan (45^\circ - 30^\circ)$$

$$\begin{aligned}
&= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} \quad \left[\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \right] \\
&= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\left(\frac{1}{\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} \\
&= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3+1-2\sqrt{3}}{(\sqrt{3})^2 - (1)^2} \\
&= \frac{4-2\sqrt{3}}{3-1} = 2-\sqrt{3}
\end{aligned}$$

Question 6:

Prove that: $\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right) - \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right) = \sin(x+y)$

Answer

$$\begin{aligned}
&\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right) - \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right) \\
&= \frac{1}{2} \left[2\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right) \right] + \frac{1}{2} \left[-2\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right) \right] \\
&= \frac{1}{2} \left[\cos\left\{\left(\frac{\pi}{4}-x\right) + \left(\frac{\pi}{4}-y\right)\right\} + \cos\left\{\left(\frac{\pi}{4}-x\right) - \left(\frac{\pi}{4}-y\right)\right\} \right] \\
&\quad + \frac{1}{2} \left[\cos\left\{\left(\frac{\pi}{4}-x\right) + \left(\frac{\pi}{4}-y\right)\right\} - \cos\left\{\left(\frac{\pi}{4}-x\right) - \left(\frac{\pi}{4}-y\right)\right\} \right] \\
&\quad \left[\begin{aligned} \because 2\cos A \cos B &= \cos(A+B) + \cos(A-B) \\ -2\sin A \sin B &= \cos(A+B) - \cos(A-B) \end{aligned} \right] \\
&= 2 \times \frac{1}{2} \left[\cos\left\{\left(\frac{\pi}{4}-x\right) + \left(\frac{\pi}{4}-y\right)\right\} \right] \\
&= \cos\left[\frac{\pi}{2} - (x+y)\right] \\
&= \sin(x+y) \\
&= \text{R.H.S}
\end{aligned}$$

Question 7:

$$\frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$$

Prove that:

Answer

It is known that $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ and $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\frac{\tan\left(\frac{\pi}{4} + x\right)}{\tan\left(\frac{\pi}{4} - x\right)} = \frac{\left(\frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4} \tan x}\right)}{\left(\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4} \tan x}\right)} = \frac{\left(\frac{1 + \tan x}{1 - \tan x}\right)}{\left(\frac{1 - \tan x}{1 + \tan x}\right)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2 = \text{R.H.S.}$$

∴ L.H.S. =

Question 8:

$$\frac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$$

Prove that

Answer

$$\begin{aligned} \text{L.H.S.} &= \frac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\frac{\pi}{2} + x\right)} \\ &= \frac{[-\cos x][\cos x]}{(\sin x)(-\sin x)} \\ &= \frac{-\cos^2 x}{-\sin^2 x} \\ &= \cot^2 x \\ &= \text{R.H.S.} \end{aligned}$$

Question 9:

$$\cos\left(\frac{3\pi}{2} + x\right) \cos(2\pi + x) \left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x) \right] = 1$$

Answer

$$\begin{aligned} \text{L.H.S.} &= \cos\left(\frac{3\pi}{2} + x\right) \cos(2\pi + x) \left[\cot\left(\frac{3\pi}{2} - x\right) + \cot(2\pi + x) \right] \\ &= \sin x \cos x [\tan x + \cot x] \\ &= \sin x \cos x \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \right) \\ &= (\sin x \cos x) \left[\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \right] \\ &= 1 = \text{R.H.S.} \end{aligned}$$

Question 10:

Prove that $\sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x = \cos x$

Answer

$$\begin{aligned} \text{L.H.S.} &= \sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x \\ &= \frac{1}{2} [2 \sin(n+1)x \sin(n+2)x + 2 \cos(n+1)x \cos(n+2)x] \\ &= \frac{1}{2} \left[\cos\{(n+1)x - (n+2)x\} - \cos\{(n+1)x + (n+2)x\} \right. \\ &\quad \left. + \cos\{(n+1)x + (n+2)x\} + \cos\{(n+1)x - (n+2)x\} \right] \\ &\quad \left[\because -2 \sin A \sin B = \cos(A+B) - \cos(A-B) \right. \\ &\quad \left. 2 \cos A \cos B = \cos(A+B) + \cos(A-B) \right] \\ &= \frac{1}{2} \times 2 \cos\{(n+1)x - (n+2)x\} \\ &= \cos(-x) = \cos x = \text{R.H.S.} \end{aligned}$$

Question 11:

Prove that $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2} \sin x$

Answer

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)$$

It is known that

$$\begin{aligned} \therefore \text{L.H.S.} &= \cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) \\ &= -2 \sin\left\{\frac{\left(\frac{3\pi}{4} + x\right) + \left(\frac{3\pi}{4} - x\right)}{2}\right\} \cdot \sin\left\{\frac{\left(\frac{3\pi}{4} + x\right) - \left(\frac{3\pi}{4} - x\right)}{2}\right\} \\ &= -2 \sin\left(\frac{3\pi}{4}\right) \sin x \\ &= -2 \sin\left(\pi - \frac{\pi}{4}\right) \sin x \\ &= -2 \sin \frac{\pi}{4} \sin x \\ &= -2 \times \frac{1}{\sqrt{2}} \times \sin x \\ &= -\sqrt{2} \sin x \\ &= \text{R.H.S.} \end{aligned}$$

Question 12:

Prove that $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$

Answer

It is known that

$$\begin{aligned} \sin A + \sin B &= 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right), \quad \sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \\ \therefore \text{L.H.S.} &= \sin^2 6x - \sin^2 4x \\ &= (\sin 6x + \sin 4x) (\sin 6x - \sin 4x) \\ &= \left[2 \sin\left(\frac{6x+4x}{2}\right) \cos\left(\frac{6x-4x}{2}\right)\right] \left[2 \cos\left(\frac{6x+4x}{2}\right) \sin\left(\frac{6x-4x}{2}\right)\right] \\ &= (2 \sin 5x \cos x) (2 \cos 5x \sin x) \\ &= (2 \sin 5x \cos 5x) (2 \sin x \cos x) \\ &= \sin 10x \sin 2x \end{aligned}$$

= R.H.S.

Question 13:

Prove that $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$

Answer

It is known that

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right), \quad \cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.} = \cos^2 2x - \cos^2 6x$$

$$= (\cos 2x + \cos 6x) (\cos 2x - \cos 6x)$$

$$= \left[2 \cos\left(\frac{2x+6x}{2}\right) \cos\left(\frac{2x-6x}{2}\right) \right] \left[-2 \sin\left(\frac{2x+6x}{2}\right) \sin\left(\frac{2x-6x}{2}\right) \right]$$

$$= [2 \cos 4x \cos(-2x)] [-2 \sin 4x \sin(-2x)]$$

$$= [2 \cos 4x \cos 2x] [-2 \sin 4x (-\sin 2x)]$$

$$= (2 \sin 4x \cos 4x) (2 \sin 2x \cos 2x)$$

$$= \sin 8x \sin 4x$$

= R.H.S.

Question 14:

Prove that $\sin 2x + 2 \sin 4x + \sin 6x = 4 \cos^2 x \sin 4x$

Answer

$$\text{L.H.S.} = \sin 2x + 2 \sin 4x + \sin 6x$$

$$= [\sin 2x + \sin 6x] + 2 \sin 4x$$

$$= \left[2 \sin\left(\frac{2x+6x}{2}\right) \cos\left(\frac{2x-6x}{2}\right) \right] + 2 \sin 4x$$

$$\left[\because \sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \right]$$

$$= 2 \sin 4x \cos(-2x) + 2 \sin 4x$$

$$= 2 \sin 4x \cos 2x + 2 \sin 4x$$

$$= 2 \sin 4x (\cos 2x + 1)$$

$$= 2 \sin 4x (2 \cos^2 x - 1 + 1)$$

$$= 2 \sin 4x (2 \cos^2 x)$$

$$= 4\cos^2 x \sin 4x$$

$$= \text{R.H.S.}$$

Question 15:

Prove that $\cot 4x (\sin 5x + \sin 3x) = \cot x (\sin 5x - \sin 3x)$

Answer

$$\text{L.H.S} = \cot 4x (\sin 5x + \sin 3x)$$

$$= \frac{\cot 4x}{\sin 4x} \left[2 \sin \left(\frac{5x+3x}{2} \right) \cos \left(\frac{5x-3x}{2} \right) \right]$$
$$\left[\because \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right]$$
$$= \left(\frac{\cos 4x}{\sin 4x} \right) [2 \sin 4x \cos x]$$

$$= 2 \cos 4x \cos x$$

$$\text{R.H.S.} = \cot x (\sin 5x - \sin 3x)$$

$$= \frac{\cos x}{\sin x} \left[2 \cos \left(\frac{5x+3x}{2} \right) \sin \left(\frac{5x-3x}{2} \right) \right]$$
$$\left[\because \sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right]$$
$$= \frac{\cos x}{\sin x} [2 \cos 4x \sin x]$$

$$= 2 \cos 4x \cdot \cos x$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Question 16:

$$\text{Prove that } \frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$$

Answer

It is known that

$$\cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right), \quad \sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right)$$

$$\begin{aligned} \therefore \text{L.H.S} &= \frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} \\ &= \frac{-2 \sin\left(\frac{9x+5x}{2}\right) \cdot \sin\left(\frac{9x-5x}{2}\right)}{2 \cos\left(\frac{17x+3x}{2}\right) \cdot \sin\left(\frac{17x-3x}{2}\right)} \\ &= \frac{-2 \sin 7x \cdot \sin 2x}{2 \cos 10x \cdot \sin 7x} \\ &= -\frac{\sin 2x}{\cos 10x} \\ &= \text{R.H.S.} \end{aligned}$$

Question 17:

Prove that $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

Answer

It is known that

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right), \quad \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.} = \frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x}$$

$$\begin{aligned} &= \frac{2 \sin\left(\frac{5x+3x}{2}\right) \cdot \cos\left(\frac{5x-3x}{2}\right)}{2 \cos\left(\frac{5x+3x}{2}\right) \cdot \cos\left(\frac{5x-3x}{2}\right)} \\ &= \frac{2 \sin 4x \cdot \cos x}{2 \cos 4x \cdot \cos x} \\ &= \frac{\sin 4x}{\cos 4x} \\ &= \tan 4x = \text{R.H.S.} \end{aligned}$$

Question 18:

Prove that $\frac{\sin x - \sin y}{\cos x + \cos y} = \tan \frac{x-y}{2}$

Answer

It is known that

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right), \quad \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.} = \frac{\sin x - \sin y}{\cos x + \cos y}$$

$$= \frac{2 \cos\left(\frac{x+y}{2}\right) \cdot \sin\left(\frac{x-y}{2}\right)}{2 \cos\left(\frac{x+y}{2}\right) \cdot \cos\left(\frac{x-y}{2}\right)}$$

$$= \frac{\sin\left(\frac{x-y}{2}\right)}{\cos\left(\frac{x-y}{2}\right)}$$

$$= \tan\left(\frac{x-y}{2}\right) = \text{R.H.S.}$$

Question 19:

$$\text{Prove that } \frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$$

Answer

It is known that

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right), \quad \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\therefore \text{L.H.S.} = \frac{\sin x + \sin 3x}{\cos x + \cos 3x}$$

$$\begin{aligned}
&= \frac{2 \sin\left(\frac{x+3x}{2}\right) \cos\left(\frac{x-3x}{2}\right)}{2 \cos\left(\frac{x+3x}{2}\right) \cos\left(\frac{x-3x}{2}\right)} \\
&= \frac{\sin 2x}{\cos 2x} \\
&= \tan 2x \\
&= \text{R.H.S}
\end{aligned}$$

Question 20:

Prove that $\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2 \sin x$

Answer

It is known that

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right), \quad \cos^2 A - \sin^2 A = \cos 2A$$

$$\therefore \text{L.H.S.} = \frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x}$$

$$= \frac{2 \cos\left(\frac{x+3x}{2}\right) \sin\left(\frac{x-3x}{2}\right)}{-\cos 2x}$$

$$= \frac{2 \cos 2x \sin(-x)}{-\cos 2x}$$

$$= -2 \times (-\sin x)$$

$$= 2 \sin x = \text{R.H.S.}$$

Question 21:

Prove that $\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

Answer

$$\text{L.H.S.} = \frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$$

$$\begin{aligned}
&= \frac{(\cos 4x + \cos 2x) + \cos 3x}{(\sin 4x + \sin 2x) + \sin 3x} \\
&= \frac{2 \cos \left(\frac{4x+2x}{2} \right) \cos \left(\frac{4x-2x}{2} \right) + \cos 3x}{2 \sin \left(\frac{4x+2x}{2} \right) \cos \left(\frac{4x-2x}{2} \right) + \sin 3x} \\
&\left[\because \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right), \sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\
&= \frac{2 \cos 3x \cos x + \cos 3x}{2 \sin 3x \cos x + \sin 3x} \\
&= \frac{\cos 3x (2 \cos x + 1)}{\sin 3x (2 \cos x + 1)} \\
&= \cot 3x = \text{R.H.S.}
\end{aligned}$$

Question 22:

Prove that $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

Answer

$$\text{L.H.S.} = \cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x$$

$$= \cot x \cot 2x - \cot 3x (\cot 2x + \cot x)$$

$$= \cot x \cot 2x - \cot (2x + x) (\cot 2x + \cot x)$$

$$= \cot x \cot 2x - \left[\frac{\cot 2x \cot x - 1}{\cot x + \cot 2x} \right] (\cot 2x + \cot x)$$

$$\left[\because \cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} \right]$$

$$= \cot x \cot 2x - (\cot 2x \cot x - 1)$$

$$= 1 = \text{R.H.S.}$$

Question 23:

$$\text{Prove that } \tan 4x = \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$$

Answer

$$\text{It is known that } \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\therefore \text{L.H.S.} = \tan 4x = \tan 2(2x)$$

$$\begin{aligned}
 &= \frac{2 \tan 2x}{1 - \tan^2(2x)} \\
 &= \frac{2 \left(\frac{2 \tan x}{1 - \tan^2 x} \right)}{1 - \left(\frac{2 \tan x}{1 - \tan^2 x} \right)^2} \\
 &= \frac{\left(\frac{4 \tan x}{1 - \tan^2 x} \right)}{\left[1 - \frac{4 \tan^2 x}{(1 - \tan^2 x)^2} \right]} \\
 &= \frac{\left(\frac{4 \tan x}{1 - \tan^2 x} \right)}{\left[\frac{(1 - \tan^2 x)^2 - 4 \tan^2 x}{(1 - \tan^2 x)^2} \right]} \\
 &= \frac{4 \tan x (1 - \tan^2 x)}{(1 - \tan^2 x)^2 - 4 \tan^2 x} \\
 &= \frac{4 \tan x (1 - \tan^2 x)}{1 + \tan^4 x - 2 \tan^2 x - 4 \tan^2 x} \\
 &= \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x} = \text{R.H.S.}
 \end{aligned}$$

Question 24:

Prove that $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$

Answer

$$\text{L.H.S.} = \cos 4x$$

$$= \cos 2(2x)$$

$$= 1 - 2 \sin^2 2x \quad [\cos 2A = 1 - 2 \sin^2 A]$$

$$= 1 - 2(2 \sin x \cos x)^2 \quad [\sin 2A = 2 \sin A \cos A]$$

$$= 1 - 8 \sin^2 x \cos^2 x$$

$$= \text{R.H.S.}$$

Question 25:

Prove that: $\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$

Answer

$$\text{L.H.S.} = \cos 6x$$

$$= \cos 3(2x)$$

$$= 4 \cos^3 2x - 3 \cos 2x [\cos 3A = 4 \cos^3 A - 3 \cos A]$$

$$= 4 [(2 \cos^2 x - 1)^3 - 3 (2 \cos^2 x - 1) [\cos 2x = 2 \cos^2 x - 1]]$$

$$= 4 [(2 \cos^2 x)^3 - (1)^3 - 3 (2 \cos^2 x)^2 + 3 (2 \cos^2 x)] - 6 \cos^2 x + 3$$

$$= 4 [8 \cos^6 x - 1 - 12 \cos^4 x + 6 \cos^2 x] - 6 \cos^2 x + 3$$

$$= 32 \cos^6 x - 4 - 48 \cos^4 x + 24 \cos^2 x - 6 \cos^2 x + 3$$

$$= 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$$

$$= \text{R.H.S.}$$

Exercise 3.4

Question 1:

Find the principal and general solutions of the equation $\tan x = \sqrt{3}$

Answer

$$\tan x = \sqrt{3}$$

It is known that $\tan \frac{\pi}{3} = \sqrt{3}$ and $\tan \left(\frac{4\pi}{3} \right) = \tan \left(\pi + \frac{\pi}{3} \right) = \tan \frac{\pi}{3} = \sqrt{3}$

Therefore, the principal solutions are $x = \frac{\pi}{3}$ and $\frac{4\pi}{3}$.

$$\text{Now, } \tan x = \tan \frac{\pi}{3}$$

$$\Rightarrow x = n\pi + \frac{\pi}{3}, \text{ where } n \in \mathbb{Z}$$

Therefore, the general solution is $x = n\pi + \frac{\pi}{3}, \text{ where } n \in \mathbb{Z}$

Question 2:

Find the principal and general solutions of the equation $\sec x = 2$

Answer

$$\sec x = 2$$

It is known that $\sec \frac{\pi}{3} = 2$ and $\sec \frac{5\pi}{3} = \sec \left(2\pi - \frac{\pi}{3} \right) = \sec \frac{\pi}{3} = 2$

Therefore, the principal solutions are $x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

$$\text{Now, } \sec x = \sec \frac{\pi}{3}$$

$$\Rightarrow \cos x = \cos \frac{\pi}{3} \quad \left[\sec x = \frac{1}{\cos x} \right]$$

$$\Rightarrow x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in \mathbb{Z}$$

Therefore, the general solution is $x = 2n\pi \pm \frac{\pi}{3}$, where $n \in Z$

Question 3:

Find the principal and general solutions of the equation $\cot x = -\sqrt{3}$

Answer

$$\cot x = -\sqrt{3}$$

It is known that $\cot \frac{\pi}{6} = \sqrt{3}$

$$\therefore \cot\left(\pi - \frac{\pi}{6}\right) = -\cot \frac{\pi}{6} = -\sqrt{3} \text{ and } \cot\left(2\pi - \frac{\pi}{6}\right) = -\cot \frac{\pi}{6} = -\sqrt{3}$$

$$\text{i.e., } \cot \frac{5\pi}{6} = -\sqrt{3} \text{ and } \cot \frac{11\pi}{6} = -\sqrt{3}$$

Therefore, the principal solutions are $x = \frac{5\pi}{6}$ and $\frac{11\pi}{6}$.

$$\text{Now, } \cot x = \cot \frac{5\pi}{6}$$

$$\Rightarrow \tan x = \tan \frac{5\pi}{6} \quad \left[\cot x = \frac{1}{\tan x} \right]$$

$$\Rightarrow x = n\pi + \frac{5\pi}{6}, \text{ where } n \in Z$$

Therefore, the general solution is $x = n\pi + \frac{5\pi}{6}$, where $n \in Z$

Question 4:

Find the general solution of $\operatorname{cosec} x = -2$

Answer

$$\operatorname{cosec} x = -2$$

It is known that

$$\operatorname{cosec} \frac{\pi}{6} = 2$$

$$\therefore \operatorname{cosec} \left(\pi + \frac{\pi}{6} \right) = -\operatorname{cosec} \frac{\pi}{6} = -2 \text{ and } \operatorname{cosec} \left(2\pi - \frac{\pi}{6} \right) = -\operatorname{cosec} \frac{\pi}{6} = -2$$

$$\text{i.e., } \operatorname{cosec} \frac{7\pi}{6} = -2 \text{ and } \operatorname{cosec} \frac{11\pi}{6} = -2$$

Therefore, the principal solutions are $x = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

$$\text{Now, } \operatorname{cosec} x = \operatorname{cosec} \frac{7\pi}{6}$$

$$\Rightarrow \sin x = \sin \frac{7\pi}{6} \quad \left[\operatorname{cosec} x = \frac{1}{\sin x} \right]$$

$$\Rightarrow x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbb{Z}$$

Therefore, the general solution is $x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbb{Z}$

Question 5:

Find the general solution of the equation $\cos 4x = \cos 2x$

Answer

$$\cos 4x = \cos 2x$$

$$\Rightarrow \cos 4x - \cos 2x = 0$$

$$\Rightarrow -2 \sin \left(\frac{4x+2x}{2} \right) \sin \left(\frac{4x-2x}{2} \right) = 0$$

$$\left[\because \cos A - \cos B = -2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right]$$

$$\Rightarrow \sin 3x \sin x = 0$$

$$\Rightarrow \sin 3x = 0 \quad \text{or} \quad \sin x = 0$$

$$\therefore 3x = n\pi \quad \text{or} \quad x = n\pi, \text{ where } n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{n\pi}{3} \quad \text{or} \quad x = n\pi, \text{ where } n \in \mathbb{Z}$$

Question 6:

Find the general solution of the equation $\cos 3x + \cos x - \cos 2x = 0$

Answer

$$\cos 3x + \cos x - \cos 2x = 0$$

$$\Rightarrow 2 \cos \left(\frac{3x+x}{2} \right) \cos \left(\frac{3x-x}{2} \right) - \cos 2x = 0 \quad \left[\cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right]$$

$$\Rightarrow 2 \cos 2x \cos x - \cos 2x = 0$$

$$\Rightarrow \cos 2x (2 \cos x - 1) = 0$$

$$\Rightarrow \cos 2x = 0 \quad \text{or} \quad 2 \cos x - 1 = 0$$

$$\Rightarrow \cos 2x = 0 \quad \text{or} \quad \cos x = \frac{1}{2}$$

$$\therefore 2x = (2n+1) \frac{\pi}{2} \quad \text{or} \quad \cos x = \cos \frac{\pi}{3}, \text{ where } n \in Z$$

$$\Rightarrow x = (2n+1) \frac{\pi}{4} \quad \text{or} \quad x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in Z$$

Question 7:

Find the general solution of the equation $\sin 2x + \cos x = 0$

Answer

$$\sin 2x + \cos x = 0$$

$$\Rightarrow 2 \sin x \cos x + \cos x = 0$$

$$\Rightarrow \cos x (2 \sin x + 1) = 0$$

$$\Rightarrow \cos x = 0 \quad \text{or} \quad 2 \sin x + 1 = 0$$

$$\text{Now, } \cos x = 0 \Rightarrow \cos x = (2n+1) \frac{\pi}{2}, \text{ where } n \in Z$$

$$2 \sin x + 1 = 0$$

$$\Rightarrow \sin x = \frac{-1}{2} = -\sin \frac{\pi}{6} = \sin \left(\pi + \frac{\pi}{6} \right) = \sin \left(\pi + \frac{\pi}{6} \right) = \sin \frac{7\pi}{6}$$

$$\Rightarrow x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in Z$$

Therefore, the general solution is $(2n+1) \frac{\pi}{2}$ or $n\pi + (-1)^n \frac{7\pi}{6}$, $n \in Z$.

Question 8:

Find the general solution of the equation $\sec^2 2x = 1 - \tan 2x$

Answer

$$\sec^2 2x = 1 - \tan 2x$$

$$\Rightarrow 1 + \tan^2 2x = 1 - \tan 2x$$

$$\Rightarrow \tan^2 2x + \tan 2x = 0$$

$$\Rightarrow \tan 2x(\tan 2x + 1) = 0$$

$$\Rightarrow \tan 2x = 0 \quad \text{or} \quad \tan 2x + 1 = 0$$

Now, $\tan 2x = 0$

$$\Rightarrow \tan 2x = \tan 0$$

$$\Rightarrow 2x = n\pi + 0, \text{ where } n \in Z$$

$$\Rightarrow x = \frac{n\pi}{2}, \text{ where } n \in Z$$

$$\tan 2x + 1 = 0$$

$$\Rightarrow \tan 2x = -1 = -\tan \frac{\pi}{4} = \tan \left(\pi - \frac{\pi}{4} \right) = \tan \frac{3\pi}{4}$$

$$\Rightarrow 2x = n\pi + \frac{3\pi}{4}, \text{ where } n \in Z$$

$$\Rightarrow x = \frac{n\pi}{2} + \frac{3\pi}{8}, \text{ where } n \in Z$$

Therefore, the general solution is $\frac{n\pi}{2}$ or $\frac{n\pi}{2} + \frac{3\pi}{8}, n \in Z$.

Question 9:

Find the general solution of the equation $\sin x + \sin 3x + \sin 5x = 0$

Answer

$$\sin x + \sin 3x + \sin 5x = 0$$

$$(\sin x + \sin 5x) + \sin 3x = 0$$

$$\Rightarrow \left[2 \sin \left(\frac{x+5x}{2} \right) \cos \left(\frac{x-5x}{2} \right) \right] + \sin 3x = 0$$

$$\left[\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right]$$

$$\Rightarrow 2 \sin 3x \cos(-2x) + \sin 3x = 0$$

$$\Rightarrow 2 \sin 3x \cos 2x + \sin 3x = 0$$

$$\Rightarrow \sin 3x (2 \cos 2x + 1) = 0$$

$$\Rightarrow \sin 3x = 0 \quad \text{or} \quad 2 \cos 2x + 1 = 0$$

Now, $\sin 3x = 0 \Rightarrow 3x = n\pi$, where $n \in \mathbb{Z}$

i.e., $x = \frac{n\pi}{3}$, where $n \in \mathbb{Z}$

$$2 \cos 2x + 1 = 0$$

$$\Rightarrow \cos 2x = \frac{-1}{2} = -\cos \frac{\pi}{3} = \cos \left(\pi - \frac{\pi}{3} \right)$$

$$\Rightarrow \cos 2x = \cos \frac{2\pi}{3}$$

$$\Rightarrow 2x = 2n\pi \pm \frac{2\pi}{3}, \text{ where } n \in \mathbb{Z}$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{3}, \text{ where } n \in \mathbb{Z}$$

Therefore, the general solution is $\frac{n\pi}{3}$ or $n\pi \pm \frac{\pi}{3}$, $n \in \mathbb{Z}$

NCERT Miscellaneous Solution

Question 1: Prove that: $2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$

Answer

L.H.S.

$$\begin{aligned} &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \left(\frac{\frac{3\pi}{13} + \frac{5\pi}{13}}{2} \right) \cos \left(\frac{\frac{3\pi}{13} - \frac{5\pi}{13}}{2} \right) \left[\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) \right] \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \left(\frac{-\pi}{13} \right) \\ &= 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + 2 \cos \frac{4\pi}{13} \cos \frac{\pi}{13} \\ &= 2 \cos \frac{\pi}{13} \left[\cos \frac{9\pi}{13} + \cos \frac{4\pi}{13} \right] \\ &= 2 \cos \frac{\pi}{13} \left[2 \cos \left(\frac{\frac{9\pi}{13} + \frac{4\pi}{13}}{2} \right) \cos \left(\frac{\frac{9\pi}{13} - \frac{4\pi}{13}}{2} \right) \right] \\ &= 2 \cos \frac{\pi}{13} \left[2 \cos \frac{\pi}{2} \cos \frac{5\pi}{26} \right] \\ &= 2 \cos \frac{\pi}{13} \times 2 \times 0 \times \cos \frac{5\pi}{26} \\ &= 0 = \text{R.H.S} \end{aligned}$$

Question 2:

Prove that: $(\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x = 0$

Answer

L.H.S.

$$= (\sin 3x + \sin x) \sin x + (\cos 3x - \cos x) \cos x$$

$$\begin{aligned}
&= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x - \cos^2 x \\
&= \cos 3x \cos x + \sin 3x \sin x - (\cos^2 x - \sin^2 x) \\
&= \cos(3x - x) - \cos 2x \quad [\cos(A - B) = \cos A \cos B + \sin A \sin B] \\
&= \cos 2x - \cos 2x \\
&= 0 \\
&= \text{R.H.S.}
\end{aligned}$$

Question 3:

Prove that: $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4 \cos^2 \frac{x+y}{2}$

Answer

$$\begin{aligned}
\text{L.H.S.} &= (\cos x + \cos y)^2 + (\sin x - \sin y)^2 \\
&= \cos^2 x + \cos^2 y + 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y \\
&= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) + 2(\cos x \cos y - \sin x \sin y) \\
&= 1 + 1 + 2 \cos(x + y) \quad [\cos(A + B) = (\cos A \cos B - \sin A \sin B)] \\
&= 2 + 2 \cos(x + y) \\
&= 2[1 + \cos(x + y)] \\
&= 2 \left[1 + 2 \cos^2 \left(\frac{x+y}{2} \right) - 1 \right] \quad [\cos 2A = 2 \cos^2 A - 1] \\
&= 4 \cos^2 \left(\frac{x+y}{2} \right) = \text{R.H.S.}
\end{aligned}$$

Question 4:

Prove that: $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \frac{x-y}{2}$

Answer

$$\text{L.H.S.} = (\cos x - \cos y)^2 + (\sin x - \sin y)^2$$

$$\begin{aligned}
&= \cos^2 x + \cos^2 y - 2 \cos x \cos y + \sin^2 x + \sin^2 y - 2 \sin x \sin y \\
&= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) - 2[\cos x \cos y + \sin x \sin y] \\
&= 1 + 1 - 2[\cos(x - y)] \quad [\cos(A - B) = \cos A \cos B + \sin A \sin B] \\
&= 2[1 - \cos(x - y)] \\
&= 2\left[1 - \left\{1 - 2\sin^2\left(\frac{x - y}{2}\right)\right\}\right] \quad [\cos 2A = 1 - 2\sin^2 A] \\
&= 4\sin^2\left(\frac{x - y}{2}\right) = \text{R.H.S.}
\end{aligned}$$

Question 5:

Prove that: $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$

Answer

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)$$

It is known that

$$\text{L.H.S.} = \sin x + \sin 3x + \sin 5x + \sin 7x$$

$$= (\sin x + \sin 5x) + (\sin 3x + \sin 7x)$$

$$= 2 \sin\left(\frac{x+5x}{2}\right) \cdot \cos\left(\frac{x-5x}{2}\right) + 2 \sin\left(\frac{3x+7x}{2}\right) \cos\left(\frac{3x-7x}{2}\right)$$

$$= 2 \sin 3x \cos(-2x) + 2 \sin 5x \cos(-2x)$$

$$= 2 \sin 3x \cos 2x + 2 \sin 5x \cos 2x$$

$$= 2 \cos 2x [\sin 3x + \sin 5x]$$

$$= 2 \cos 2x \left[2 \sin\left(\frac{3x+5x}{2}\right) \cdot \cos\left(\frac{3x-5x}{2}\right) \right]$$

$$= 2 \cos 2x [2 \sin 4x \cdot \cos(-x)]$$

$$= 4 \cos 2x \sin 4x \cos x = \text{R.H.S.}$$

Question 6:

$$\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$$

Prove that:

Answer

It is known that

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right), \quad \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right)$$

$$\text{L.H.S.} = \frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)}$$

$$= \frac{\left[2 \sin\left(\frac{7x+5x}{2}\right) \cdot \cos\left(\frac{7x-5x}{2}\right)\right] + \left[2 \sin\left(\frac{9x+3x}{2}\right) \cdot \cos\left(\frac{9x-3x}{2}\right)\right]}{\left[2 \cos\left(\frac{7x+5x}{2}\right) \cdot \cos\left(\frac{7x-5x}{2}\right)\right] + \left[2 \cos\left(\frac{9x+3x}{2}\right) \cdot \cos\left(\frac{9x-3x}{2}\right)\right]}$$

$$= \frac{[2 \sin 6x \cdot \cos x] + [2 \sin 6x \cdot \cos 3x]}{[2 \cos 6x \cdot \cos x] + [2 \cos 6x \cdot \cos 3x]}$$

$$= \frac{2 \sin 6x [\cos x + \cos 3x]}{2 \cos 6x [\cos x + \cos 3x]}$$

$$= \tan 6x$$

$$= \text{R.H.S.}$$

Question 7:

Prove that: $\sin 3x + \sin 2x - \sin x = 4 \sin x \cos \frac{x}{2} \cos \frac{3x}{2}$

Answer

$$\text{L.H.S.} = \sin 3x + \sin 2x - \sin x$$

$$\begin{aligned}
&= \sin 3x + (\sin 2x - \sin x) \\
&= \sin 3x + \left[2 \cos \left(\frac{2x+x}{2} \right) \sin \left(\frac{2x-x}{2} \right) \right] \quad \left[\sin A - \sin B = 2 \cos \left(\frac{A+B}{2} \right) \sin \left(\frac{A-B}{2} \right) \right] \\
&= \sin 3x + \left[2 \cos \left(\frac{3x}{2} \right) \sin \left(\frac{x}{2} \right) \right] \\
&= \sin 3x + 2 \cos \frac{3x}{2} \sin \frac{x}{2} \\
&= 2 \sin \frac{3x}{2} \cdot \cos \frac{3x}{2} + 2 \cos \frac{3x}{2} \sin \frac{x}{2} \quad [\sin 2A = 2 \sin A \cdot \cos B] \\
&= 2 \cos \left(\frac{3x}{2} \right) \left[\sin \left(\frac{3x}{2} \right) + \sin \left(\frac{x}{2} \right) \right] \\
&= 2 \cos \left(\frac{3x}{2} \right) \left[2 \sin \left\{ \frac{\left(\frac{3x}{2} \right) + \left(\frac{x}{2} \right)}{2} \right\} \cos \left\{ \frac{\left(\frac{3x}{2} \right) - \left(\frac{x}{2} \right)}{2} \right\} \right] \quad \left[\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right] \\
&= 2 \cos \left(\frac{3x}{2} \right) \cdot 2 \sin x \cos \left(\frac{x}{2} \right) \\
&= 4 \sin x \cos \left(\frac{x}{2} \right) \cos \left(\frac{3x}{2} \right) = \text{R.H.S.}
\end{aligned}$$

Question 8:

$$\tan x = -\frac{4}{3}, x \text{ in quadrant II}$$

Answer

Here, x is in quadrant II.

$$\text{i.e., } \frac{\pi}{2} < x < \pi$$

$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Therefore, $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are all positive.

It is given that $\tan x = -\frac{4}{3}$.

$$\sec^2 x = 1 + \tan^2 x = 1 + \left(-\frac{4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}$$

$$\therefore \cos^2 x = \frac{9}{25}$$

$$\Rightarrow \cos x = \pm \frac{3}{5}$$

As x is in quadrant II, $\cos x$ is negative.

$$\square \quad \cos x = \frac{-3}{5}$$

$$\text{Now, } \cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$\Rightarrow \frac{-3}{5} = 2 \cos^2 \frac{x}{2} - 1$$

$$\Rightarrow 2 \cos^2 \frac{x}{2} = 1 - \frac{3}{5}$$

$$\Rightarrow 2 \cos^2 \frac{x}{2} = \frac{2}{5}$$

$$\Rightarrow \cos^2 \frac{x}{2} = \frac{1}{5}$$

$$\Rightarrow \cos \frac{x}{2} = \frac{1}{\sqrt{5}}$$

$\left[\because \cos \frac{x}{2} \text{ is positive} \right]$

$$\therefore \cos \frac{x}{2} = \frac{\sqrt{5}}{5}$$

$$\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1$$

$$\Rightarrow \sin^2 \frac{x}{2} + \left(\frac{1}{\sqrt{5}}\right)^2 = 1$$

$$\Rightarrow \sin^2 \frac{x}{2} = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\Rightarrow \sin \frac{x}{2} = \frac{2}{\sqrt{5}} \quad \left[\because \sin \frac{x}{2} \text{ is positive} \right]$$

$$\therefore \sin \frac{x}{2} = \frac{2\sqrt{5}}{5}$$

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{2}{\sqrt{5}} \right)}{\left(\frac{1}{\sqrt{5}} \right)} = 2$$

Thus, the respective values of $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are $\frac{2\sqrt{5}}{5}$, $\frac{\sqrt{5}}{5}$, and 2.

Question 9:

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ for $\cos x = -\frac{1}{3}$, x in quadrant III

Answer

Here, x is in quadrant III.

$$\text{i.e., } \pi < x < \frac{3\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$$

Therefore, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are negative, whereas $\sin \frac{x}{2}$ is positive.

It is given that $\cos x = -\frac{1}{3}$.

$$\cos x = 1 - 2\sin^2 \frac{x}{2}$$

$$\Rightarrow \sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$

$$\Rightarrow \sin^2 \frac{x}{2} = \frac{1 - \left(-\frac{1}{3} \right)}{2} = \frac{\left(1 + \frac{1}{3} \right)}{2} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

$$\Rightarrow \sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \quad \left[\because \sin \frac{x}{2} \text{ is positive} \right]$$

$$\therefore \sin \frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$\text{Now, } \cos x = 2 \cos^2 \frac{x}{2} - 1$$

$$\Rightarrow \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} = \frac{1 + \left(-\frac{1}{3}\right)}{2} = \frac{\left(\frac{3-1}{3}\right)}{2} = \frac{\left(\frac{2}{3}\right)}{2} = \frac{1}{3}$$

$$\Rightarrow \cos \frac{x}{2} = -\frac{1}{\sqrt{3}} \quad \left[\because \cos \frac{x}{2} \text{ is negative} \right]$$

$$\therefore \cos \frac{x}{2} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{\left(-\frac{1}{\sqrt{3}}\right)} = -\sqrt{2}$$

Thus, the respective values of $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are $\frac{\sqrt{6}}{3}$, $-\frac{\sqrt{3}}{3}$, and $-\sqrt{2}$.

Question 10:

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ for $\sin x = \frac{1}{4}$, x in quadrant II

Answer

Here, x is in quadrant II.

$$\text{i.e., } \frac{\pi}{2} < x < \pi$$

$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Therefore, $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, and $\tan \frac{x}{2}$ are all positive.

It is given that $\sin x = \frac{1}{4}$.

$$\cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\Rightarrow \cos x = -\frac{\sqrt{15}}{4} \quad [\cos x \text{ is negative in quadrant II}]$$

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} = \frac{1 - \left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4 + \sqrt{15}}{8}$$

$$\Rightarrow \sin \frac{x}{2} = \sqrt{\frac{4 + \sqrt{15}}{8}} \quad \left[\because \sin \frac{x}{2} \text{ is positive} \right]$$

$$= \sqrt{\frac{4 + \sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \sqrt{\frac{8 + 2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8 + 2\sqrt{15}}}{4}$$

$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2} = \frac{1 + \left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4 - \sqrt{15}}{8}$$

$$\Rightarrow \cos \frac{x}{2} = \sqrt{\frac{4 - \sqrt{15}}{8}} \quad \left[\because \cos \frac{x}{2} \text{ is positive} \right]$$

$$= \sqrt{\frac{4 - \sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \sqrt{\frac{8 - 2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8 - 2\sqrt{15}}}{4}$$

$$\begin{aligned}\tan \frac{x}{2} &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4}\right)}{\left(\frac{\sqrt{8-2\sqrt{15}}}{4}\right)} = \frac{\sqrt{8+2\sqrt{15}}}{\sqrt{8-2\sqrt{15}}} \\ &= \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}}} \\ &= \sqrt{\frac{(8+2\sqrt{15})^2}{64-60}} = \frac{8+2\sqrt{15}}{2} = 4+\sqrt{15}\end{aligned}$$

Thus, the respective values of $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ are $\frac{\sqrt{8+2\sqrt{15}}}{4}$, $\frac{\sqrt{8-2\sqrt{15}}}{4}$,
and $4+\sqrt{15}$