## trignometry

## 1. Introduction

Trigonometry (from Greek trigõnon, "triangle" and metron, "measure") is a branch of mathematics that studies relationships involving lengths and angles of triangles. The field emerged during the $3^{\text {rd }}$ century BC from applications of geometry to astronomical studies.
Trigonometry is most simply associated with planar right angle triangles (each of which is a two-dimensional triangle with one angle equal to 90 degrees). The applicability to non-right-angle triangles exists, but, since any non-right-angle triangle (on a flat plane) can be bisected to create two right-angle triangles, most problems can be reduced to calculations on right-angle triangles. Thus the majority of applications relate to right-angle triangles

## What is angle

An angle which has its vertex at the origin and one side lying on the positive $x$-axis. It can have a measure which positive or negative and can be greater than $360^{\circ}$
If the direction of rotation is anticlockwise, angle is positive. If the direction of rotation is clockwise, angle is negative
Once you have made a full circle $\left(360^{\circ}\right)$ keep going and you will see that the angle is greater than $360^{\circ}$. In fact you can go around as many times as you like. The same thing happens when you go clockwise. The negative angle just keeps on increasing
It can be measured in degrees or radian

## Degree and Radian

They both are unit of measurement of angles
Radian: A unit of measure for angles. One radian is the angle made at the center of a circle by an arc whose length is equal to the radius of the circle.
Degree: If a rotation from the initial side to terminal side is $(1 / 360)$ of a revolution, the angle is said to have a measure of one degree, written as $1^{\circ}$. A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as $1^{\prime \prime}$, and one sixtieth of a minute is called a second, written as $1^{\prime}$.
Thus, $1^{\circ}=60$ ', $1^{\prime}=60 "$

## Relation between Degree and Radian

$2 \pi$ radian $=360^{\circ} \pi$ radian $=180^{\circ} 1$ radian $=(180 / \pi)^{\circ}$

| Degree | $30^{\circ}$ | $45^{\circ}$ | $60^{\circ}$ | $90^{\circ}$ | $120^{\circ}$ | $180^{\circ}$ | $360^{\circ}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Radian | $\pi / 6$ | $\pi / 4$ | $\pi / 3$ | $\pi / 2$ | $2 \pi / 3$ | $\pi$ | $2 \pi$ |

## Trigonmetric Ratio's

In a right angle triangle ABC where $\mathrm{B}=90^{\circ}$, we can define six ratio's for the two sides i.e Hypotenuse/Base,
Base/Perpendicular,Perpendicular/Base,Base/hypotenuse,Hypotenuse/Perpendicular,Perpedicular/Hyptenuse
Trignometric ratio's are defined as
$\sin \theta=$ Perpendicular/Hypotenuse
$\operatorname{cosec} \theta=$ Hypotenuse/Perpendicular
$\cos \theta=$ Base/Hypotenuse
$\sec \theta=$ Hypotenuse/Base
$\tan \theta=$ Perpendicular/Base
$\cot \theta=$ Base/Perpendicular
Notice that each ratio in the right-hand column is the inverse, or the reciprocal, of the ratio in the left-hand column.
The reciprocal of $\sin \theta$ is $\csc \theta$; and vice-versa.
The reciprocal of $\cos \theta$ is $\sec \theta$.
And the reciprocal of $\tan \theta$ is $\cot \theta$
These are valid for acute angles.
We are now going to define them for any angles and they are called now the Trigometric functions.

## Trignometric functions:

Consider a unit circle with center at the origin $O$ and Let $P$ be any point on the circle with $P(a, b)$. And let call the angle $x$ We use the coordinates of $P$ to define the cosine of the angle and the sine of the angle. Specifically, the $x$-coordinate of $B$ is the cosine of the angle, and the $y$-coordinate of $B$ is the sine of the angle. Also it is clear

$$
\begin{aligned}
& a^{2}+b^{2}=1 \\
& \cos ^{2} x+\sin ^{2} x=1
\end{aligned}
$$



## Properties of these functions

Sine and cosine are periodic functions of period $\$ 360^{\wedge}\{\backslash \mathrm{circ}\} \$$, that is, of period $\$ 2 \backslash$ pi $\$$. That's because sines and cosines are defined in terms of angles, and you can add multiples of $\$ 360^{\wedge}\{1 \mathrm{circ}\} \$$, or $\$ 2 \backslash p i \$$, and it doesn't change the angle. Thus, for any angle $x$
$\sin (x+2 \pi)=\sin (x)$ and $\cos (x+2 \pi)=\cos (x)$
or we can say that
$\sin (2 n \pi+x)=\sin x, \$ n \operatorname{lin} Z \$, \cos (2 n \pi+x)=\cos x, \$ n \operatorname{lin} Z \$$
Where $Z$ is the set of all integers
$\sin \mathrm{x}=0$ implies $\mathrm{x}=\mathrm{n} \pi$, where n is any integer
$\cos x=0$ implies $x=(2 n+1)(\pi / 2)$
The other trignometric function are defined as
$\operatorname{cosec}(x)=1 / \sin (x)$ where $x \neq n \pi$, where $n$ is any integer
$\sec (x)=1 / \cos (x)$ where $x \neq(2 n+1)(\pi / 2)$ where $n$ is any integer
$\tan (x)=\sin (x) / \cos (x)$ where $x \neq(2 n+1)(\pi / 2)$ where $n$ is any integer
$\cot (x)=\cos (x) / \sin (x)$ where $x \neq n \pi$, where n is any integer
For all real $x$
$\sin ^{2}(x)+\cos ^{2}(x)=1$
$1+\tan ^{2}(\mathrm{x})=\sec ^{2}(\mathrm{x})$
$1+\cot ^{2}(x)=\operatorname{cosec}^{2}(x)$

## What is is Odd function and Even Function

We have come across these adjectives 'odd' and 'even' when applied to functions, but it's important to know them. A function $f$ is said to be an odd function
if for any number $x, f(-x)=-f(x)$.
A function $f$ is said to be an even function if for any number $x, f(-x)=f(x)$.
Many functions are neither odd nor even functions, but some of the most important functions are one or the other.

## Example:

Any polynomial with only odd degree terms is an odd function, for example, $f(x)=2 x^{7}+9 x^{5}-x$. (Note that all the powers of $x$ are odd numbers.)
Similarly, any polynomial with only even degree terms is an even function. For example, $f(x)=6 x^{8}-6 x^{2}-5$.
Based on above defination we can call Sine is an odd function, and cosine is even
$\sin (-x)=-\sin x$, and
$\cos (-x)=\cos x$.
These facts follow from the symmetry of the unit circle across the $x$-axis. The angle $-x$ is the same angle as $x$ except it's on the other side of the x -axis. Flipping a point ( $\mathrm{x}, \mathrm{y}$ ) to the other side of the x -axis makes it into ( $\mathrm{x},-$ $y$ ), so the $y$-coordinate is negated, that is, the sine is negated, but the $x$-coordinate remains the same, that is, the cosine is unchanged.
Now since in unit circle
$-1 \leq \mathrm{a} \leq 1$
$-1 \leq b \leq 1$
It follows that for all $x$

$$
-1 \leq \sin (x) \leq 1
$$

$-1 \leq \cos (x) \leq 1$ Also We know from previous classes,
a,b are both positive in Ist quadrant i.e $0<x<\pi / 2$ It implies that $\sin$ is positive and cos is postive

$$
a \text { is negative and } b \text { is positive in IInd quadrant } i . e \pi / 2<x<\pi / t \text { implies that } \sin \text { is negative and cos is postive }
$$ $a$ and $b$ both are negative in III quadrant ie. $\pi<x<3 \pi / 2$ It implies that sin is negative and cos is negative $a$ is positive and $b$ is negative in IV quadrant $\mathrm{i}, \mathrm{e} 3 \pi / 2<x<2 \pi$ It implies that $\sin$ is positive and cos is negative Similarly sign can be obtained for other functions

## Domain and Range of trigonimetric functions

$y=f(x)=\operatorname{Sin}(x)$

Domain : It is defined for all real values of $x$
Range : $-1 \leq y \leq 1$
Period:2m
It is a odd function

$y=f(x)=\cos (x)$
Domain : It is defined for all real values of $x$
Range : $-1 \leq \mathrm{y} \leq 1$
Period:2ד
It is even function


## ) $y=f(x)=\tan (x)$

Domain : It is defined for all real values of $x$ except $x \neq(2 n+1)(\pi / 2)$ where $n$ is any Range : All the real numbers

Period: $\pi$
It is a odd function

$y=f(x)=\cot (x)$
Domain : It is defined for all real values of $x$ except $x \neq n \pi$, where $n$ is any integer
Range : All the real numbers
Period:п
It is a odd function


$$
y=f(x)=\sec (x)
$$

Domain : It is defined for all real values of $x$ except $x \neq(2 n+1)(\pi / 2)$ where $n$ is any integer
Range : $(-\infty,-1] \square[1, \infty)$
Period:2ா
It is even function

$y=f(x)=\operatorname{cosec}(x)$
Domain :It is defined for all real values of $x$ except $x \neq n \pi$, where $n$ is any integer

## Range : $(-\infty,-1] \square[1, \infty)$

Period:2ா
It is odd function


## Trigonometrics function of Sum and difference of angles

## Sin and cos function

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cos(A+B)=\operatorname{cos}(A)\operatorname{cos}(B)-\operatorname{sin}(A)\operatorname{sin}(B)
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$\cos (A-B)=\cos (A) \cos (B)+\sin (A) \sin (B)$
$\cos (\pi / 2-A)=\sin (A)$
$\sin (\pi / 2-A)=\cos (A)$
$\sin (A+B)=\sin (A) \cos (B)+\sin (B) \cos (A)$
$\sin (A-B)=\sin (A) \cos (B)-\sin (B) \cos (A)$

Similary we can have defined other sin and cos sum and differences

## Tan and cot functions

If none of the angles $x, y$ and $(x+y)$ is an odd multiple of $\pi / 2$
$\$ \tan (A+B)=\backslash f r a c\{\tan (A)+\tan (B)\}\{1-\tan (A) \tan (B)\} \$$
$\$ \tan (A-B)=\backslash \operatorname{frac}\{\tan (A)-\tan (B)\}\{1+\tan (A) \tan (B)\} \$$
If none of the angles $x, y$ and $(x+y)$ is an multiple of $\pi$
$\$ \cot (A+B)=\backslash \operatorname{frac}\{\cot (A) \cot (B)-1\}\{\cot (A)+\cot (B)\} \$$
$\$ \cot (A-B)=\backslash \operatorname{frac}\{\cot (A) \cot (B)+1\}\{\cot (B)-\cot (A)\} \$$
Now lets explore the multiple of $x$. These all can be proved from above equations
Double of $x$
$\$ \cos 2 x=\cos ^{\wedge}\{\wedge\{2\}\} x-\sin ^{\wedge}\{\wedge\{2\}\} x=2 \cos ^{\wedge}\{\wedge\{2\}\} x-1=1-2 \sin ^{\wedge}\{\wedge\{2\}\} x=\mid f r a c\{1-\tan \wedge\{\wedge\{2\}\} x\}\{1+\tan \wedge\{\wedge\{2\}\} x\} \$$
$\$ \sin 2 x=2 \cos (x) \sin (x)=\backslash f r a c\{2 \tan (x)\}\{1+\tan \wedge\{\wedge\{2\}\} x\} \$$
$\$ \tan 2 x=\mid f r a c\{2 \tan (x)\}\{1-\tan \wedge\{\wedge\{2\}\} x\} \$$
Triple of $x$
$\$ \sin 3 x=3 \sin (x)-4 \sin ^{\wedge}\{3\} x \$$
$\$ \cos 3 x=4 \cos ^{\wedge}\{3\} x-3 \cos (x) \$$
$\$ \tan (3 x)=\mid f r a c\left\{3 \tan x-\tan ^{\wedge}\{\wedge\{3\}\} x\right\}\{1-3 \tan \wedge\{\wedge\{2\}\} x\} \$$
Some other Important functions
$\$ \cos (A)+\cos (B)=2 \cos \backslash f r a c\{A+B\}\{2\} \cos \backslash f r a c\{A-B\}\{2\} \$$

## $\$ \cos (A)-\cos (B)=-2 \sin \backslash f r a c\{A+B\}\{2\} \sin \backslash f r a c\{A-B\}\{2\} \$$

$\$ \sin (A)+\sin (B)=2 \sin \backslash f r a c\{A+B\}\{2\} \cos \backslash f r a c\{A-B\}\{2\} \$$ $\$ \sin (A)-\sin (B)=2 \cos \backslash f r a c\{A+B\}\{2\} \sin \backslash f r a c\{A-B\}\{2\} \$$

## Trigonometric equations

Equation involving trigonometric function defined above are call trigonemtric equation. Since we know that value of sin and cos function repeat after $2 \pi$ interval and value of tan function repeat after $\pi$ interval. So there will be infinite solution for these equations. We define two forms of solutions here
Principle Solution: The solution in the rangle $0 \leq x \leq 2 \pi$
General Solution: The expression involving integer n which gives all solutions of a trigonometric equation is called the general solution.
We shall use 'Z' to denote the set of integers.

## Some Important points in that regard

(a)
$\sin \mathrm{x}=0$ implies $\mathrm{x}=\mathrm{n} \pi$, where n is any integer
$\cos x=0$ implies $x=(2 n+1)(\pi / 2)$
(b)
$\sin x=\sin y$ then $x=n \pi+(-1)^{n} y$ where $n$ is any integer
$\cos x=\cos y$ then $x=2 n \pi+y$ or $x=2 n \pi-y$ where $n$ is any integer
$\tan \mathrm{x}=\operatorname{tany}$ then $\mathrm{x}=\mathrm{n} \pi+\mathrm{y}$ or $\mathrm{x}=\mathrm{n} \pi-\mathrm{y}$ where n is any integer

## Some basics Tips to solve the trigonometry questions

1) Always try to bring the multiple angles to single angles using basic formla.Make sure all your angles are the same. Using $\sin (2 X)$ and $\sin X$ is difficult, but if you use $\sin 2 X=2 \sin (x) \cos (x)$, that leaves $\sin (x)$ and $\cos (x)$, and now all your functions match.
The same goes for addition and subtraction: don't try working with $\sin (X+Y)$ and $\sin X$. Instead, use $\sin (X+Y)=$ $\sin (x) \cos (y)+\cos (x) \sin (y)$ so that all the angles match
2) Converting to sin and cos all the items in the problem using basic formula. I have mentioned sin and cos as they are easy to solve. You can use any other also.
3) Check all the angles for sums and differences and use the appropriate identities to remove them.
4) Use phythagorean identifies to simplfy the equations
5) Practice and Practice. You will soon start figuring out the equation and there symmetry to resolve them fast

## Chapter 3: Trigonometric functions

## Concept:

Radian measure- relation between degree and radian- trigonometric functions- sign of trigonometric functions- trigonometric functions of sum and difference of two angles- trigonometric equations- sine formula- cosine formula- their applications.

Notes:

- If in a circle of radius r , an arc of length ' 1 ' subtends an angle of $\theta$ radians then $1=r \theta$.
- Radian measure $=(\pi / 180) \mathrm{x}$ degree measure.
- $\operatorname{Sin}(-x)=-\sin x$
- $\operatorname{Cos}(-x)=\cos x$
- $\operatorname{Cos}(2 n \pi+x)=\cos x$
- $\operatorname{Sin}(2 n \pi+x)=\sin x$
- $\operatorname{Sin} x=0$ gives $x=n \pi$ where $n \in \mathbf{Z}$
- Cos $x=0$ gives $x=(2 n+1) \pi / 2$ where $n \in \mathbf{Z}$
- Refer text book for other formulas.


## Text book questions

Ex:3.1
Ex:3.2
Ex:3.3 18, 21*,

Ex:3.4
Misc. Ex:
Examples:

Questions: $1^{*}, 2^{*}, 3^{*}, 6$
Questions: 6, 7, 8, 9, 10
Questions: 5, 6, $7^{*}, 11,12^{*}, 14^{*}, 15^{*}, 16$,

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22^{* *}, 23^{* *}, 24^{*}, 25^{*}
$$

Questions: 5, 6, 7, 8, 9**
Questions: 2, 3, 5, 6, 7, $8^{*}, 9^{*}, 10^{*}$
Questions: $24^{* *}, 25^{* *}, 26^{*}, 27^{*}, 29^{* *}$
Examples:

$$
\text { Questions: } 27^{* *}, 28^{*}
$$

## Extra/ HOT Questions

1. The angles of a triangle are in A.P and the greatest angle is double the least. Express the angles in degrees and radians
2. Show that the equation $\operatorname{cosec} x=4 a b /(a+b)^{2}(a b>0)$ is possible if $a=b$
3. Show that a) $\sin 150 \cos 120+\cos 330 \sin 660=-1$

$$
\text { b) } \frac{\cos (90+x) \sec (-x) \tan (180-x)}{\sec (360-x) \sin (180+x) \cot (90=x)}=1
$$

4. If $\tan \mathrm{x}=\frac{m}{m+1}$ and $\tan y=\frac{1}{2 m+1}$, show that $\mathrm{x}+\mathrm{y}=45^{0}$
5. Show that the following:

> a) $\cos 10 \cos 50 \cos 60 \cos 70=3 / 16$
> b) $\sin 10 \sin 50 \sin 60 \sin 70=\sqrt{3} / 16$
c) $\cos 20 \cos 40 \cos 60=1 / 8$
6. If $\sin x \sin y=1 / 4$ and $3 \tan x=4 \tan y$ then prove that $\sin (x+y)=7 / 16$
7. Prove that $\frac{\sin 11 x \sin x+\sin 7 x \sin 3 x}{\cos 11 x \sin x+\cos 7 x \sin 3 x}=\tan 8 x$
8. If $\mathrm{m} \tan (\mathrm{x}-30)=\mathrm{n} \tan (\mathrm{x}+120)$ then show that $\frac{m-n}{2(m+n)}=\frac{1}{4} \sec 2 x$
9. Solve the equation $4 \sin x \cos x+2 \sin x+2 \cos x+1=0$
10. Solve the triangle when $\mathrm{c}=3.4 \mathrm{~cm}, \mathrm{~A}=25^{\circ}, \mathrm{B}=85^{\circ}$ [ans; $a=1.53 \mathrm{~cm}, b=3.6 \mathrm{~cm}, \mathrm{C}=80^{\circ}$
11. Show that for any parallelogram, if a and $b$ are the sides of two non parallel sides, x is the angle between these two sides and d is the length of the diagonal that has a common vertex with sides $a$ and $b$ ,then $d^{2}=a^{2}+b^{2}+2 a b \cos x$

