Introduction of Complex Numbers,Properties of

Complex Number

Complex numbers are the numbers of the form a+ib where $i=\sqrt{(-1)}$ and a and b are real numbers.

Definition:- Complex numbers are defined as an ordered pair of real numbers like (x,y) where

$z=(x,y)=x+iy$

and both x and y are real numbers and x is known as real part of complex number and y is known as imaginary part of the complex number.

Example

$z_1= 2+4i$

$z_1= 8-4i$

Properties of Complex Number

Addition of complex numbers

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ then

$z_1+z_2=(x_1+x_2)+i(y_1+y_2)$

Subtraction

$z_1-z_2=(x_1-x_2)+i(y_1-y_2)$

Multiplication

$(z_1.z_2)=(x_1+iy_1).(x_2+iy_2)$

Multiplicative Inverse

for z=x+iy

z^{-1} is given by

=$(\frac{a}{a^{2}+b^{2}})+i(\frac{-b}{a^{2}+b^{2}})$

Division

To divide complex number by another , first write quotient as a fraction. Then reduce the denominator complex number to multiplicative Inverse and then simple multiplication applies

Example

Find the value of

(1+i)/(1+2i)

Solution:

The multiplicative inverse of (1+2i) is given

as

=$(\frac{a}{a^{2}+b^{2}})+i(\frac{-b}{a^{2}+b^{2}})$

=(1/5-2i/5)

So (1+i)/(1+2i)

=(1+i)(1/5-2i/5)

=1/5-2i/5+ i/5+2/5

=-i/5 +3/5

Conjugate of Complex Numbers

Let z be the complex number defined as

$z=x+iy$

The conjugate of z is defined as

$\bar{z}=x-iy$

So by defination of Conjugate of any complex number is obtained by replacing i with -i

Properties of Conjugate Number

For $z=x+iy$

1) $\bar(\bar{z})=z$

2) $z+\bar{z}=2x$

3) $z-\bar{z}=2iy$

4) if $\bar{z}=z$ then it means z is real number

5) if $z+\bar{z}=0$ then it means z is pure imaginary number

6) $z\bar{z}=(x^{2}+y^{2})$

7) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

8) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

9) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

10) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ when z_{2} is not zero

Example:

z=2-3i

Then

$\bar{z}=2+3i$

Modulus Of complex Number

The module of a complex number

$z=x+iy$

is defined as |z|

|z|=$\sqrt{(x^2+y^2)}$

Clearly $|z| \geqslant 0$

Properties of Module of Complex Number

For $z=x+iy$

1) $|z|=0$ then it mean $x=y=0$

2) $|z|=|\bar{z}|=|-z|$

3) $z\bar{z}=|z|^{2}$

4) $|z_{1}z_{2}|=|z_{1}||z_{2}|$

5) $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2\,Real(z_1\overline{z_2})$

6) $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\,Real(z_1\overline{z_2})$

7) $|\frac{z_{1}}{z_{2}}|=\frac{|z_{1}|}{|z_{2}|}$ when z_{2} is not zero

The proof of the above properties are quite self explanatory

Example

z=3-4i

Then

|z|=5

Reciprocal or Multiplicative Inverse of Complex Number using Complex Conjugate

z= x+iy

Then

$\frac{1}{z}=\frac{\bar{z}}{|z|^{2}}$

Examples

1)(3 + i)(1 + 7i)

(3 + i)(1 + 7i)= 3×1 + 3×7i + i×1+ i×7i

= 3 + 21i + i + $7i^2$

= 3 + 21i + i - 7 (because i^2 = -1)

= -11 + 22i

We can generalized this multiplication

(a+bi)(x+ui) = (ax-by) + (ay+bx)i

2) Solving the equation

(1+2i)z=(1-i)

Solution:

$z=\frac{1-i}{1+2i}$

Dividing the conjugate for denominator

$z=[\frac{1-i}{1+2i}][\frac{1-2i}{|1-2i}]$

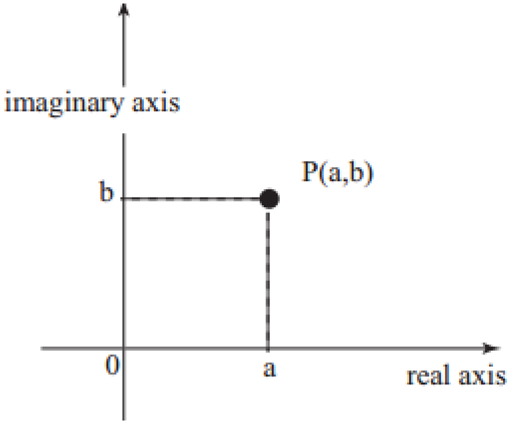
$z==[\frac{1-2i-i+2}{5}]$

$z==[\frac{3-i}{5}]$

Graphical Representation of Complex Number

Complex number z=a+ib can be represented by a point P(a,b) on a xy plane. The x -axis represent the real part while the y-axis represent the imgainary part

Such a diagram is called Argand diagram or Complex plane



Points to Note

1) The length OP is called the module of z and is denoted by |z|

|z|=$\sqrt{(x^2+y^2)}$

OP=$\sqrt{(x^2+y^2)}$

2) Purely real number lies on X-axis while purely imaginary number lies on y-axis

3) The Line OP makes an angle θ with the positive direction of x-axis in anti-clockwise sense is called the argument or amplitude of z

It is given by

$\theta=tan^{-1}\frac{y}{x}$

4) Argument of a complex number is not unique since if θ is the value of argument then $2n\pi + \theta$ (n=0, ± 1, ± 2, ...) are also values of the argument. Thus, argument of complex number can have infinite number of values which differ from each other by any multiple of 2π

5) The unique value of θ such that $\pi < \theta <= \pi$ is called the principal value of the amplitude or principal argument

6) The principal argument of the complex number is find using the below steps

Step 1) for z=a+ib , find the acute angle value of $\theta=tan^{-1}|\frac{y}{x}|$

Step 2) Look for the values of a ,b

if (a,b) lies in First quadrant then Argument=θ

if (a,b) lies in second quadrant then Argument =$\pi-\theta$

if (a,b) lies in third quadrant then Argument =$-\pi+\theta$

if (a,b) lies in Fourth quadrant then Argument =$-\theta$

7) The conjugate of the complex number lies symmetrically about the x-axis

8) All the real number can be represent as complex number with zero imaginary part. like a +i(0) . So we can term real numbers as subset of bigger set of complex numbers

Vector Representation of the complex number

Just like a vector,A complex number on the argand plane for two things modulus and arg(z) which is direction.

So all the complex number can be defined as position vector on the argand plane.

The addition ,substraction of complex number can happen just like vector addition and substraction

Polar Representation of the complex number

We can represent complex number is terms of polar coordinates

Let r be any non negative number and θ any real number. If we take $x=rcos\theta$ and $y=rsin\theta$ then, $r=\sqrt{x^2+y^2}$which is the modulus of z and $\theta=tan^{-1}\frac{y}{x}$ which is the argument or amplitude of z and is denoted by arg.z

we also have $x+iy=r(cos\theta+isin\theta)=r[cos(2n\pi+\theta)+isin(2n\pi+\theta)]$, where n=0, $\pm $1, $\pm $2, ...

The unique value of θ such that $\pi < \theta <= \pi $ is called the principal value of the amplitude or principal argument

So we will be writing the polar coordinates form of complex number in principal argument form

How to convert any complex number in Polar form

Step 1) for z=a+ib , find the acute angle value of $\theta=tan^{-1}|\frac{y}{x}|$

Step 2) Look for the values of a ,b

if (a,b) lies in First quadrant then Argument=θ

if (a,b) lies in second quadrant then Argument =$\pi-\theta$

if (a,b) lies in third quadrant then Argument =$-\pi+\theta$

if (a,b) lies in Fourth quadrant then Argument =$-\theta$

Step 3) Polar form = |z|[cos (arg) + sin (arg)]

Rotation of Complex Number

Multiplying i is a rotation by 90 degrees counter-clockwise

Multiplying by -i is a rotation of 90 degrees clockwise

<u>Example</u>

z=1

If we multiply it by i, it becomes

z=i so that it has rotated by the angle 90 degrees

What is the significance of Complex Numbers? Why they are required?

Real numbers such as natural number,rational number , irrational number are invented in the history as and when we encounter various mathematical needs. Same happen with the complex numbers.

We had no solution for the problem

$x^2=-1$

Eular was the first mathematicain to introduce solution to this problem,he introduced the symbol i

$ i=\sqrt{(-1)}$

So $i^2=-1$

So solution to the problem becomes

x=i or -i

He called the symbol i as imaginary unit.

Just like all the other number ,this number was added to our Number vocabulary. This like other numbers is useful in explaining where physical explanation.

It is very useful in the field Electrical and electronics.

Examples:

1. Find the modulus and amplitude for the complex number

z=-1-i

Solution:

We have already given the steps for modulus and arg

Modulus

|z|=$\sqrt{(x^2+y^2)}$

How to find the arg

Step 1) for z=a+ib , find the acute angle value of $\theta=tan^{-1}|\frac{y}{x}|$

Step 2) Look for the values of a ,b

if (a,b) lies in First quadrant then Argument=θ

if (a,b) lies in second quadrant then Argument =$\pi-\theta$

if (a,b) lies in third quadrant then Argument =$-\pi+\theta$

if (a,b) lies in Fourth quadrant then Argument =$-\theta$

So

|z|=$\sqrt{2}$

Acute angle

$\theta=tan^{-1}|\frac{y}{x}|$

$\theta=\frac{\pi}{4}$

Now the complex lies in third quadrant

So $arg=\frac{-3\pi}{4}$

2) Find the polar coordinate equation for the above complex number

Solution: we know the polar coordinate equation is given by

Polar form = |z|[cos (arg) + sin (arg)]

so

$z=

$$\sqrt{2}[\cos\left(\frac{-3\pi}{4}\right)+\sin\left(\frac{-3\pi}{4}\right)]$$

Flashback of Quadratic equation From previous Classes

Quadratic Polynomial

$P(x) = ax^2 + bx + c$ where $a \neq 0$

Quadratic equation

$ax^2 + bx + c = 0$ where $a \neq 0$

Solution or root of the Quadratic equation

A real number α is called the root or solution of the quadratic equation if

$a\alpha^2 + b\alpha + c = 0$

Some other points to remember

The root of the quadratic equation is the zeroes of the polynomial p(x).

We know from chapter two that a polynomial of degree can have max two zeroes. So a quadratic equation can have maximum two roots

A quadratic equation has no real roots if $b^2 - 4ac < 0$

How to Solve Quadratic equation

S.no	Method	Working
		This method we factorize the equation by splitting the middle term b In $ax^2+bx+c=0$

1	factorization	Example $6x^2-x-2=0$ 1) First we need to multiple the coefficient a and c.In this case =6X-2=-12 2) Splitting the middle term so that multiplication is 12 and difference is the coefficient b $6x^2+3x-4x-2=0$ $3x(2x+1)-2(2x+1)=0$ $(3x-2)(2x+1)=0$ 3) Roots of the equation can be find equating the factors to zero $3x-2=0 \Rightarrow x=3/2$ $2x+1=0 \Rightarrow x=-1/2$
2	Square method	In this method we create square on LHS and RHS and then find the value. $ax^2+bx+c=0$ 1) $x^2+(b/a)x+(c/a)=0$ 2) $(x+b/2a)^2-(b/2a)^2+(c/a)=0$ 3) $(x+b/2a)^2=(b^2-4ac)/4a^2$ 4) Example $x^2+4x-5=0$ 1) $(x+2)^2-4-5=0$ 2) $(x+2)^2=9$ 3) Roots of the equation can be find using square root on both the sides $x+2=-3 \Rightarrow x=-5$ $x+2=3 \Rightarrow x=1$
3	Quadratic method	For quadratic equation $ax^2+bx+c=0$, roots are given by $x=\frac{-b+\sqrt{b^{2}-4ac}}{2a}$ and $x=\frac{-b-\sqrt{b^{2}-4ac}}{2a}$

		For $b^2 -4ac > 0$, Quadratic equation has two real roots of different value For $b^2-4ac =0$, quadratic equation has one real root For $b^2-4ac < 0$, no real roots for quadratic equation

Nature of roots of Quadratic equation

S.no	Condition	Nature of roots
1	$b^2 -4ac > 0$	Two distinct real roots
2	$b^2-4ac =0$	One real root
3	$b^2-4ac < 0$	No real roots

Solved examples

1) Find the roots of the quadratic equation

$x^2 -6x=0$

Solution

There is no constant term in this quadratic equation, we can x as common factor

$x(x-6)=0$

So roots are $x=0$ and $x=6$

2) Solve the quadratic equation

$x^2 -16=0$

Solution

$x^2 -16=0$

$x^2 =16$

or $x=4$ or -4

3) Solve the quadratic equation by factorization method

$x^2 -x -20=0$

Solution

1) First we need to multiple the coefficient a and c.In this case =1X-20=-20

The possible multiple are 4,5 ,2,10

2) The multiple 4,5 suite the equation

$x^2-5x+4x-20=0$

$x(x-5)+4(x-5)=0$

$(x+4)(x-5)=0$

or $x=-4$ or 5

4) Solve the quadratic equation by Quadratic method

$x^2-3x-18=0$

Solution

For quadratic equation

$ax^2+bx+c=0$,

roots are given by

$x=\frac{-b+\sqrt{b^{2}-4ac}}{2a}$

and

$x=\frac{-b-\sqrt{b^{2}-4ac}}{2a}$

Here $a=1$

$b=-3$

$c=-18$

Substituting these values,we get

$x=6$ and -3

Introduction of Complex roots

We have already studied about Complex number in previous chapter,Now it times to use in quadratic equation

We know that if

$b^2-4ac < 0$

We dont have real roots

Now if $b^2-4ac < 0$

then $4ac-b^2 > 0$

So now we can define imaginary roots of the equation as

$k_{1}=\frac{-b+i\sqrt{4ac-b^{2}}}{2a}$

$k_{2}=\frac{-b-i\sqrt{4ac-b^{2}}}{2a}$

Complex roots occurs in conjugate pairs and it is very clear from the roots given above

The quadratic equations containing complex roots can be solved using the Factorization method,square method and quadratic method explained above

Complex Quadratic quation

So far we have read about quadratic equation where the coefficent are real. Complex quadratic equation are the equations where the coefficent are complex numbers.

These quadratic equation can also be solved using Factorization method,square method and quadratic method explained above

The roots may not conjugate pair in these quadratic equations

Examples

1) Solve the quadratic equation

$x^2-2x+10=0$

Solution

$x^2-2x+1+9=0$

$(x-1)^2 - (3i)^2 = 0$

$(x-1-3i)(x-1+3i)=0$

So roots are

$(1+3i)$ and $(1-3i)$

We can also obtain the same things using quadratic methods

$k_{1}=\frac{-b+i\sqrt{4ac-b^{2}}}{2a}$

$k_{2}=\frac{-b-i\sqrt{4ac-b^{2}}}{2a}$

$k_{1}=\frac{2+i\sqrt{40-2^{2}}}{2}$=1+3i

$k_{1}=\frac{2-i\sqrt{40-2^{2}}}{2}$=1-3i

2) For which values of k does the polynomial x^2+4x+k have two complex conjugate roots?

Solution:

For the roots to be complex conjugate ,we should have

$b^2 - 4ac < 0$

$16-4k < 0$

or $k > 4$

Chapter 5
COMPLEX NUMBERS AND QUADRATIC EQUATIONS

INTRODUCTION

$\sqrt{-36}$, $\sqrt{-25}$ etc do not have values in the system of real numbers.

So we need to extend the real numbers system to a larger system.

Let us denote $\sqrt{-1}$ by the symbol i.

ie $i^2 = -1$

A number of the form a+ib where a&b are real numbers is defined to be a complex number.

Eg 2+i3, $-7+\sqrt{2}\,i$, $\sqrt{3}\,i$, $4+\underline{1}i$, 5=5+0i, -7=-7+0i etc

For z = 2+i5, Re z =2 (real part)

and Im z = 5 (imaginary part)

Refer algebra of complex numbers of text book pg 98

1) Addition of complex numbers

$$(2+i3) + (-3+i2) = (2+-3) + i(3+2) = -1+5i$$

2) Difference of complex numbers

$$(2+i3)-(-3+i2) = (2+3) + i(3-2) = 5 + i$$

3) Multiplication of two complex numbers

$$\begin{aligned}(2+i3)(-3+i2) &= 2(-3+i2) + i3(-3+i2)\\ &= -6+4i-9i+6i^2\\ &= -6-5i-6 \qquad (i^2 = -1)\\ &= -12-5i\end{aligned}$$

4) Division of complex numbers

$$\begin{aligned}\frac{2+i3}{-3+i2} &= \frac{(2+i3)}{(-3+i2)} \times \frac{(-3-i2)}{(-3-i2)}\\ &= \frac{-6-4i-9i-6i^2}{(-3)^2-(i2)^2}\\ &= \frac{-6-13i+6}{9-(-1)\times 4}\\ &= \frac{-13i}{13} = -i\end{aligned}$$

5) **Equality of 2 complex numbers**
   a+ib = c+id, iff a=c & b=d
6) a+ib =0, iff a=0 and b=0
   Refer : the square roots of a negative real no & identities (text page 100,101)

Formulas

a) IF Z=a+ib then modulus of Z ie $|Z| = (a^2+b^2)^{1/2}$
b) Conjugate of Z is a-ib
c) **Multiplicative inverse of a+ib** = $\dfrac{a}{(a^2+b^2)} - \dfrac{ib}{(a^2+b^2)}$

****d) Polar representation of a complex number**

$$\boxed{a+ib = r(\cos\theta + i\sin\theta)}$$

Where r = $|Z| = (a^2+b^2)^{1/2}$ and θ = arg Z(argument or amplitude of Z which has many different values but when $-\pi < \theta \le \pi$, θ is called principal argument of Z.

Trick method to find θ

Step 1First find angle using the following

1) Cosθ = 1 and sinθ = 0 then angle = 0
2) Cosθ = 0 and sinθ = 1 then angle = π/2
3) Sinθ= $\sqrt{3}/2$ and cosθ=1/2 then angle = π/3
4) Sinθ = ½ and cosθ = $\sqrt{3}/2$ then angle = π/6

Step 2: **To find θ**

1) If both sinθ and cosθ are positive then θ = angle (first quadrant)
2) If sinθ positive, cosθ negative then θ = π-angle (second quadrant)
3) If both sinθ and cosθ are negative the θ= π+angle (third quadrant)
4) If sinθ negative and cosθ positive then θ= 2π-angle (fourth quadrant)
   Or θ = - (angle) since sin (– θ) = -sinθ and cos (–θ) = cos θ
5) If sinθ = 0 and cosθ = -1 then θ=π

****e) Formula needed to find square root of a complex number**

$(a+b)^2 = (a-b)^2 + 4ab$

ie $[x^2 + y^2]^2 = [x^2 - y^2]^2 + 4x^2y^2$

e) Powers of i

i)$\boldsymbol{i^{4k} = 1}$

ii)$\boldsymbol{i^{4k+1} = i}$

iii)$\boldsymbol{i^{4k+2} = -1}$

iv) $\boldsymbol{i^{4k+3} = -i}$**, for any integer k**

Examples:

$i^1 = i, i^2 = -1, i^3 = -i$ and $i^4 = 1$&

$$i^{19} = i^{16} \times i^3 = 1 \times -i = -i$$

g) Solutions of quadratic equation $ax^2+bx + c = 0$ with real coefficients a,b,c and $a \neq 0$ are given by $\boldsymbol{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$, If $b^2 - 4ac \geq 0$

If $b^2 - 4ac < 0$ then $\boldsymbol{x = \frac{-b \pm \sqrt{4ac-b^2}\ i}{2a}}$

Refer text page 102 the modulus and conjugate of a complex number properties given in the end. (i) to (v)
Ex 5.1
Q. 3*(1 mark), 8* (4 marks), 11**, 12**, 13**, 14**(4 Marks)

Polar form (very important)

Ex 5.2

Q 2**) Express Z = $-\sqrt{3}+i$ in the polar form and also write the modulus and the argument of Z

Solution Let $-\sqrt{3}+i = r(\cos\theta + i\sin\theta)$

Here $a = -\sqrt{3}$, $b = 1$

$r = (a^2+b^2)^{1/2} = \sqrt{3+1} = \sqrt{4} = 2$

$-\sqrt{3}+i = 2\cos\theta + i \times 2\sin\theta$

Therefore $2\cos\theta = -\sqrt{3}$ and $2\sin\theta = 1$

$\cos\theta = -\sqrt{3}/2$ and $\sin\theta = ½$

Here $\cos\theta$ negative and $\sin\theta$ positive

Therefore $\theta = \pi - \pi/6 = 5\pi/6$ (see trick method given above)

Therefore polar form of $Z = -\sqrt{3}+i = 2(\text{Cos}5\pi/6 + i\text{Sin}5\pi/6)$

$|Z| = 2$ and argument of $Z = 5\pi/6$ and $-\sqrt{3} + i = 2(Cos5\pi/6 + iSin5\pi/6)$

Ex 5.2

Q (1 to 8)** Note: Q 1) $\theta = 4\pi/3$ or principal argument $\theta = 4\pi/3 - 2\pi = -2\pi/3$

Q 5) $\theta = 5\pi/4$ or principal argument $\theta = 5\pi/4 - 2\pi = -3\pi/4$

eg 7**, eg 8**

Ex 5.3

Q 1,8,9,10 (1 mark)

Misc examples (12 to 16)**

Misc exercise

Q 4**,5**,10**,11**,12**,13**,14**,15**,16**,17*,20**

Supplementary material

eg 12**

Ex 5.4

Q (1 to 6)**

EXTRA/HOT QUESTIONS

1** Find the square roots of the following complex numbers (4 marks)

i. $6 + 8i$
ii. $3 - 4i$
iii. $2 + 3i$ (HOT)
iv. $7 - 30\sqrt{2}i$
v. $\dfrac{3 + 4i}{3 - 4i}$ (HOT)

2** Convert the following complex numbers in the polar form

i. $3\sqrt{3} + 3i$
ii. $\dfrac{1 - i}{1 + i}$

iii. 1 + i

iv. -1 + $\sqrt{3}i$

v. -3 + 3i

vi. -2 –i

3. If a+ ib =$\frac{x+i}{x-i}$ where x is a real, prove that $a^2 + b^2 = 1$ and b/a = 2x/(x^2-1) 4marks

4 Find the real and imaginary part of i. (1 mark)

5 Compute : i + i^2 + i^3+ i^4 (1 mark)

6 Solve the following quadratic equations (I mark)

i) x^2-($\sqrt{2}$ + 1)x +$\sqrt{2}$ = 0

ii) $2x^2$ + 5 = 0

7 Find the complex conjugate and multiplicative inverse of (4 mark)

i) $(2 - 5i)^2$

ii) $\frac{2 + 3i}{3 - 7i}$

8 If $|Z|$ = 2 and arg Z = π/4 then Z= _______. (1 mark)

Answers

1) i) $2\sqrt{2} + \sqrt{2}i$, $-2\sqrt{2} - \sqrt{2}i$

ii) 2 –i, -2 + i

iii) $\frac{\sqrt{\sqrt{13}+2}}{\sqrt{2}} + \frac{\sqrt{\sqrt{13}-2}\ i}{\sqrt{2}}$, $\frac{\sqrt{\sqrt{13}+2}}{\sqrt{2}} + \frac{\sqrt{\sqrt{13}-2}\ i}{\sqrt{2}}$,

iv) $5 - 3\sqrt{2}\,i$, $-5 + 3\sqrt{2}\,i$

v) 3/5 + 4/5 i, -3/5 -4/5 i

2) i) 6(cos π/6 + isinπ/6)

ii) cos (–π/2) + isin(-π/2)

iii) $\sqrt{2}$(cos π/4 + isinπ/4)

iv) 2(cos2π/3 + isin 2π/3)

iv) $3\sqrt{2}$ (cos 3π/4 + isin 3π/4)

vi) $2\sqrt{2}(\cos 5\pi/4 + i\sin 5\pi/4)$ or $2\sqrt{2}[\cos(-3\pi/4) + i\sin(-3\pi/4)]$

4) 0,1

5) 0

6) i) $\sqrt{2}$, 1

ii) $\sqrt{\frac{5}{2}}\,i$, $-\sqrt{\frac{5}{2}}\,i$

7) i) $-21 + 10i$, $\frac{-21}{541} - \frac{10}{541}i$

ii) $-\frac{15}{58} - \frac{23i}{58}$, $\frac{3 - 7i}{2 + 3i}$

8) $\sqrt{2} + i\sqrt{2}$