Permutation notes

What is factorial?

The product of first 'n' natural numbers is denoted by n!.

$n! = n(n-1)(n-2) \ldots\ldots\ldots\ldots\ldots\ldots 3.2.1.$

Example

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Note $0! = 1$

Proof $n! = n, (n-1)!$

Or $(n-1)! = [n \times (n-1)!]/n = n!/n$

Putting n = 1, we have

$0! = 1!/1$

or $0! = 1$

Example

$(n+2)! = 2550 \times n!$

Solution:

$(n+2)! = 2550 \times n!$

$(n+2)(n+1)n! = 2550 \times n!$

Canceling n!

$(n+2)(n+1) = 2550$

N=49

Fundamental Principle of counting

Addition rule : If an job can be performed in 'n' ways, & another job can be performed in 'm' ways then either of the two jobs can be performed in (m+n) ways. This rule can be extended to any finite number of jobs.

Example: In a class 12 , 20 students are from Maths streams while 30 students are from Biology stream. Teacher has to choose 1 student either from Maths stream or Biology stream to represent the class

What ways the selection can be made?

Solution:

20 students are from Maths:

Students can be chosen from Maths stream in 20 ways

30 students are from Biology stream

Students can be chosen from Biology stream in 30 ways

So by the Fundamental Principle of addition rule, the selection of 1 student from either of two streams can be done (20+30)=50 ways

Multiplication Rule : If a work can be done in m ways, another work can be done in 'n' ways, then both of the operations can be performed in m x n ways. It can be extended to any finite number of operations

Example: In a class 12 , 20 students are from Maths streams while 30 students are from Biology stream. Teacher has to choose 2 student one from Maths stream and one from Biology stream to represent the class in Quiz competition

What ways the selection can be made?

Solution:

20 students are from Maths:

1 Students can be chosen from Maths stream in 20 ways

30 students are from Biology stream

1Students can be chosen from Biology stream in 30 ways

So by the Fundamental Principle of Multiplication rule, the selection of 1 student from Maths streams and 1 students from Biology stream can be done (20X30)=600 ways

Permutation

Permutation means arrangement of things. The word arrangement is used, if the order of things is considered

Theorem 1:

Number of permutations of 'n' different things taken 'r' at a time is given by:-

^nP_r = n! / (n-r)!

Proof: Say we have 'n' different things a_1 , a_2 ...... , a_n

Clearly the first place can be filled up in 'n' ways. Number of things left after filling-up the first place = n-1

So the second-place can be filled-up in (n-1) ways. Now number of things left after filling-up the first and second places = n – 2

Now the third place can be filled-up in (n-2) ways.

Thus number of ways of filling-up first-place = n

Number of ways of filling-up second-place = n-1

Number of ways of filling-up third-place = n-2

Number of ways of filling-up r-th place = n – (r-1) = n-r+1

By multiplication – rule of counting, total no. of ways of filling up, first, second -- rth-place together :-

n (n-1) (n-2) ------------ (n-r+1)

Hence:

^nP_r = n (n-1)(n-2) --------------(n-r+1)

= [n(n-1)(n-2)----------(n-r+1)] [(n-r)(n-r-1)-----3.2.1.] / [(n-r)(n-r-1)] ----3.2.1

nPr = n!/(n-r)!

Theorem 2: Number of permutations of 'n' different things taken all at a time is given by:-

^nP_n = n!

Proof :

Now we have 'n' objects, and n-places.

Number of ways of filling-up first-place = n

Number of ways of filling-up second-place = n-1

Number of ways of filling-up third-place = n-2

Number of ways of filling-up r-th place, i.e. last place =1

Number of ways of filling-up first, second, --- n th place

= n (n-1) (n-2) ------ 2.1.

^nP_n= n!

To prove

0! = 1

We have

nPr = n!/(n-r)!

Putting r = n, we have :-

^nP_n = n! / (n-n)!= n!/0!

But ^nP_n = n!

Clearly it is possible, only when 0! = 1

Hence it is proof that 0! = 1

Example How many different signals can be made by 4 flags from 6-flags of different colours?

Solution :

Number of ways taking 4 flags out of 6-flags = $^6P_4$ = 6!/(6-4)!

= 6 x 5 x 4 x 3 = 360

Example How many words can be made by using the letters of the word "ZEBRA" taken all at a time?

Solution . There are '5' different letters of the word "ZEBRA"

Number of Permutations taking all the letters at a time = $^5P_5$

= 5! = 5X4X3X2X1=120

Restricted – Permutations

Theorem Number of permutations of 'n' things, taken 'r' at a time, when a particular thing is to be always included in each arrangement

= r $^{n-1}P_{r-1}$

Proof:

When one particular thing is always taken, then that can arranged in r ways

Now we have to arrange the remaining r-1 places with n-1 objects,so permutations will be $^{n-1}P_{r-1}$. Now by principle of counting ,both of these can be arrange in r $^{n-1}P_{r-1}$

Theorem: Number of permutations of 'n' things, taken 'r' at a time, when a particular thing is fixed:

= $^{n-1}P_{r-1}$

Proof:

When one particular thing is fixed ,then that is the only arrangement for that.Now we have to arrange the remaining r-1 places with n-1 objects,so permutations will be $^{n-1}P_{r-1}$. Now by principle of counting ,both of these can be arrange in $^{n-1}P_{r-1}$

Theorem: Number of permutations of 'n' things, taken 'r' at a time, when a particular thing is never taken

= $^{n-1}P_r$

Proof:

When one particular thing is never taken then we have to arrange the r- places with n-1 objects,so permutations will be $^{n-1}P_r$.

Another Theorem of Permutation

Theorem: There are n subjects of which p1 are alike of one kind; p2 are alike of another kind; p3 are alike of third kind and so on and pr are alike of rth kind,

such that (p1 + p2 + ... pr) = n.

Then, number of permutations of these n objects is =n!/p1! p2!.... pr!

Example: How many words can be formed with the letters of the word ‘TIGER’ when:

(i) ‘T’ and ‘R’ occupying end places.

(ii) ‘G’ being always in the middle

(iii) Vowels occupying odd-places

(iv) Vowels being never together.

v) Find the number of words formed with out any restriction

Solution

(i) When ‘T’ and ‘R’ occupying end-places

IGE. (TR)

Here (TR) are fixed, hence I,E,G can be arranged in 3! ways

But (T,R) can be arranged themselves is 2! ways.

Total number of words = 3! x 2! = 12 ways.

(ii) When ‘G’ is fixed in the middle

T.I.(G), E.R

Hence four-letter T.I.E.R. can be arranged in 4! i.e 24 ways.

(iii) Two vowels (I,E) can be arranged in the odd-places (1st, 3rd and 5th) = 3! ways.

And three consonants (T,G,R) can be arranged in the remaining places = 3 ! ways

Total number of ways= 3! x 3! = 36 ways.

(iv) Total number of words = 5! = 120

If all the vowels come together, then we have: (I.E.), T,R,G

These can be arranged in 4! ways.

But (I,E.) can be arranged themselves in 2! ways.

Number of ways, when vowels come-together = 4! x 2!

= 48 ways

Number of ways, when vowels being never-together

= 120-48 = 72 ways.

v) Total number of words = 5! = 120

Example:

Find the numbers of words formed by permuting all the letters of the following words

1) INDIA
2) RUSSIA
3) AUGUST
4) ARRANGE

Solution

1) In INDIA, There are 2 I's and all other are distinct

So Number of words formed= n!/p1! p2!.... pr!

=5!/2! =60 words

2) In RUSSIA, There are 2 S's and all other are distinct

So Number of words formed= n!/p1! p2!.... pr!

=6!/2! =360 words

3) In AUGUST, There are 2 U's and all other are distinct

So Number of words formed= n!/p1! p2!.... pr!

=6!/2! =360 words

4) In ARRANGE, There are 2 R's and 2A's and all other are distinct

So Number of words formed= n!/p1! p2!.... pr!

=7!/2!. 2!= =1260 words

Combination

We have studied about arrangement i.e permutation in the previous chapter
We learnt that arrangement of r objects taken from n objects is given by
^nP_r =n!/(n-r)!
We would be discussing the combination in this chapter
Combination is defined as the different selection made by taking some objects out of many objects irrespective of their arrangement
So arrangement does not matter in Combination while it matters in Permutation

Example

We have three letter XYZ,
We have to select two letters and arrange it
Different arrangement would be
XY YZ ZX YX,ZY,XZ

So this is permuatation

Now if we have to just select two letter and do not worry about the arrangement then
XY, YZ ,ZX

This is combination

Combination generally happens with committee selection problems as it does not matter how your committee is arranged. In other words, it generally does not matter whether you think about your committee as Vijay, Arun and Nita or as Nita, Vijay and Arun. In either case, the same three people are in the committee. Contrast this with permutations examples in previous where arrangement matters

Combination Formula

A formula for the number of possible combinations of *r* objects from a set of *n* objects. This is written in any of the ways shown below

^nC_r =n!/r! (n-r)!

Proof:

Let x be the combination of n distinct objects taken r at a time.
Now lets us consider one of the x ways. There are r objects in this one way which can be arranged in r! ways .
Similarly all these x ways can be arranged in this manner
So permutation of the n things taken r at a time will be given as
^nP_r =x r!
or x= n!/r! (n-r)!
^nC_r =n!/r! (n-r)!

Example:

Eleven students put their names on slips of paper inside a box. Three names are going to be taken out. How many different ways can the three names be chosen?

Solution

As the arrangement does not matter, so different ways are
^nC_r =n!/r! (n-r)!

$^{11}C_3$ =11!/3! (11-3)!=165

Properties of Combination Formula

^nC_r =n!/r! (n-r)!

1	^nC_n = 1 , $^nC_0$ = 1 as ^nC_n =n!/n! (n-n)!= n!/0! n!=1
2	for 0≤ r ≤ n , we have ^nC_r = $^nC_{n-r}$ As ^nC_r =n!/r! (n-r)! $^nC_{n-r}$= n!/(n-r)! (n-n+r)!= n!/r! (n-r)!
3	^nC_x = ^nC_y Then x=y or x+y=n
4	^nC_r + $^nC_{r-1}$= $^{n+1}C_r$ This property is called Pascal law's
5.	for 1≤ r ≤ n , we have ^nC_r =(n/r) $^{n-1}C_{r-1}$

6.	for $1 \le r \le n$, we have $n\,{}^{n-1}C_{r-1} = (n-r+1)\,{}^{n}C_{r-1}$

Example:

${}^{2n}C_3 : {}^{n}C_3 = 11:1$

Find the value of n

Solution:

Writing this in factorial form

$$\frac{\dfrac{2n!}{(2n-3)!\ \ 3!}}{\dfrac{n!}{(n-3)!\,3!}} = \frac{11}{1}$$

Simplifying it we get

$$\frac{2n(2n-1)(2n-2)}{n(n-1)(n-2)} = \frac{11}{1}$$

Solving this

N=6

Example:

Over the weekend, your family is going on vacation, and your mom is letting you bring your favorite video game console as well as five of your games. How many ways can you choose the five games if you have 10 games in all?

Solution

Applying combination formula

${}^{10}C_r = n!/r!\ (n-r)!$

${}^{10}C_5 = 10!/5!\ (10-5)! = 10*9*8*6/5*4*3*2*1 = 36$

Example:

In how many ways can a T20 eleven can be chosen out of batch of 15 players if

There is no restriction on the selection

A particular player is always chosen

A particular player is never chosen

Solution:

If 11players to be chosen from 15 ,then

${}^{15}C_{11} = 1365$

If the particular player is always choosen,then we need to choose 10 players from remaining 14 players

${}^{14}C_{10} = 1001$

If the particular player is never choosen,then we need to choose 11 players from remaining 14 players

$^{14}C_{11} = 364$

CHAPTER 7

PERMUTATIONS (Arrangements) AND COMBINATIONS (selections)

In permutation **order is important**, since 27 & 72 are different numbers(arrangements). In combination order is not important.

- **Fundamental principle of counting (FPC)**

Station A	m ways	Station B	n ways	Station C

then by FPC there are mn ways to go from station A to station C

- The number of permutations of n different things taken r at a time, where repetition is not allowed is given by $^nP_r = n(n-1)(n-2)\ldots\ldots(n-r+1)$ **where** $0<r\leq n$.

  eg $^5P_2 = 5 \times 4 = 20$

  $^7P_3 = 7 \times 6 \times 5 = 210$

- Factorial notation: **n! = 1 x 2 x 3 x …..x n, where n is a natural number**

  eg $5! = 1 \times 2 \times 3 \times 4 \times 5$

  we define 0! = 1

  also $n! = n(n-1)!$

  $= n(n-1)(n-2)!$

- $^nP_r = \dfrac{n!}{(n-r)!}$ **Where** $0 \leq r \leq n$

- Number of permutations of n different things, taken r at a time, where repetition is allowed is n^r

- Number of permutations of n objects taken all at a time, where P_1 objects are of first kind, P_2 objects are of second kind…..P_k objects are of the k^{th} kind and rest, if any, are all different is $\dfrac{n!}{P_1!.P_2!\ldots P_k!}$ **(eg 9)**

- The number of combinations of n different things taken r at a time is given by

  $^nC_r = \dfrac{n(n-1)(n-2)\ldots.(n-r+1)}{1.2.3\ldots\ldots r}$, **where** $0< r\leq n$

  eg $^5C_3 = \dfrac{5 \times 4 \times 3}{1 \times 2 \times 3} = {}^5C_2$

- $^{n}C_{r} = {}^{n}C_{n-r}$
  eg $^{5}C_{3} = {}^{5}C_{2}$
  $^{7}C_{5} = {}^{7}C_{2}$
- $^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$, **where** $0 \le r \le n$.
- $^{n}C_{r} = {}^{n}C_{s}$ **implies r = s or n = r+s** (eg 17*) 1 mark
- $^{n}C_{n} = {}^{n}C_{0} = 1$
- $^{n}C_{1} = n$
  eg $^{5}C_{1} = 5$
- $^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

Ex 7.1

1, 2, 4

Ex 7.2

4* , 5* (1 mark)

eg 8* (1 mark), eg 11*, 12**,13**,14**,16** (4 marks)

Ex 7.3

7*, 8*, 9**,10**,11**

Theorm 6 to prove (4 marks)*

eg 17* (1 mark) use direct formula n = 9+8 = 17 since $^{n}C_{r} = {}^{n}C_{s}$ implies r = s or n = r+s

eg 19**

Ex 7.4

2**,3*,5*,6*,7**,8*,9*

eg 21**, eg 23*(HOT), eg 24*

Misc Ex

1**,2**,3**,4*,5*,7**,10**,11**

EXTRA/HOT QUESTIONS

1) How many permutations can be made with letters of the word MATHEMATICS ? In how many of them vowels are together?
2) In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together. (HOT)
3) How many numbers greater than 56000 can be formed by using the digits 4,5,6,7,8; no digit being repeated in any number.
4) Find the number of ways in which letters of the word ARRANGEMENT can be arranged so that the two A's and two R's do not occur together. (HOT)
5) If C(2n,3): C(n,3):: 11:1 find n.
6) If P(11,r) = P(12,r-1) find r.

7) Five books, one each in Physics, Chemistry, Mathematics, English and Hindi are to be arranged on a shelf. In how many ways can this be done?
8) If $^{n}P_{r} = {}^{n}P_{r+1}$ and $^{n}C_{r} = {}^{n}C_{r-1}$ find the values of n and r.
9) A box contains five red balls and six black balls. In how many ways can six balls be selected so that there are at least two balls of each color.
10) A group consist of 4 girls and 7 boys in how many ways can a committee of five members be selected if the committee has i) no girl
    ii) atleast 1 boy and 1 girl
    iii) atlest 3 girls.

Note : atleast means $\geq$

Answers

1) 4989600, 120960
2) 282240 Hint (consider the best and the worst paper as one paper)
3) 90
4) 1678320
5) 6
6) 9
7) 120
8) n = 3, r = 2
9) 425
10) i) 21
    ii) 441
    iii) 91