# Binomial Theorem

$(x+a) = C_0 \times a + C_1 \times a + C_2 \times a + \ldots\ldots\ldots\ldots + C_r \times a \ldots\ldots + C_n \times a$

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k$$

**Proof:**

We can prove this theorem with the help of mathematical induction

Let us assume P(n) be the statement is

$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots\ldots\ldots\ldots + {}^nC_r\, x^{n-r} a^r \ldots\ldots + {}^nC_n\, x^0 a^n$

**Step 1**

Now the value of P(1)

$(x+a)^1 = {}^1C_0\, x^1 a^0 + {}^1C_1\, x^{1-1} a^1$

$=(x+a)$

So P(1) is true

**Step 2**

Now the value of P(m)

$(x+a)^m = {}^mC_0\, x^m a^0 + {}^mC_1\, x^{m-1} a^1 + {}^mC_2\, x^{m-2} a^2 + \ldots\ldots\ldots\ldots + {}^mC_r\, x^{m-r} a^r \ldots\ldots + {}^mC_n\, x^0 a^m$

Now we have to prove

$(x+a)^{m+1} = {}^{m+1}C_0\, x^{m+1} a^0 + {}^{m+1}C_1\, x^m a^1 + {}^{m+1}C_2\, x^{m-1} a^2 + \ldots\ldots\ldots\ldots + {}^{m+1}C_r\, x^{m-r-1} a^r \ldots\ldots + {}^{m+1}C_{m+1}\, x^0 a^{m+1}$

Now

$(x+a)^{m+1} = (x+a)(x+a)^m$

$= (x+a)({}^mC_0\, x^m a^0 + {}^mC_1\, x^{m-1} a^1 + {}^mC_2\, x^{m-2} a^2 + \ldots\ldots\ldots\ldots + {}^mC_r\, x^{m-r} a^r \ldots\ldots + {}^mC_n\, x^0 a^m)$

$= {}^mC_0\, x^{m+1} a^0 + ({}^mC_1 + {}^mC_0)\, x^m a^1 + ({}^mC_2 + {}^mC_1)\, x^{m-1} a^2 + \ldots\ldots\ldots$

$+({}^mC_{m-1} + {}^mC_m)\, x^1 a^m + {}^mC_m\, x^0 a^{m+1}$

As ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

So

$= {}^{m+1}C_0\, x^{m+1} a^0 + {}^{m+1}C_1\, x^m a^1 + {}^{m+1}C_2\, x^{m-1} a^2 + \ldots\ldots\ldots\ldots + {}^{m+1}C_r\, x^{m-r-1} a^r \ldots\ldots + {}^{m+1}C_{m+1}\, x^0 a^{m+1}$

So by principle of Mathematical induction, P(n) is true for n? N

## Important conclusion from Binomial Theorem

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k$$

We can easily see that $(x+a)^n$ has (n+1) terms as k can have values from 0 to n

The sum of indices of x and a in each is equal to n

$x^{n-k} a^k$

The coefficient ${}^nC_r$ is each term is called binomial coefficient

$(x-a)^n$ can be treated as $[x+(-a)]^n$

So

$$(x-a)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n-k} a^k$$

So the terms in the expansion are alternatively positive and negative. The last term is positive or negative depending on the values of n

$(1+x)^n$ can be treated as $(x+a)^n$ where x=1 and a=x
So

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

This is the expansion is ascending order of power of x

$(1+x)^n$ can be treated as $(x+a)^n$ where x=x and a=1
So

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}$$

This is the expansion is descending order of power of x

$(1-x)^n$ can be treated as $(x+a)^n$ where x=1 and a=-x
So

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

This is the expansion is ascending order of power of x

**Examples**: Expand using Binomial Theorem

$(x^2 +2)^5$

**Solution: We know from binomial Theorem**

$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots\ldots\ldots\ldots + {}^nC_r\, x^{n-r} a^r \ldots\ldots + {}^nC_n\, x^0 a^n$

So putting values $x=x^2$ ,a=2 and n=5

We get

$(x^2 +2)^5 = {}^5C_0\, (x^2)^5 + {}^5C_1\, (x^2)^4 2^1 + {}^5C_2\, (x^2)^3 2^2 + {}^5C_3\, (x^2)^2 2^3$

$+{}^5C_4\, (x^2)^1 2^4 + {}^5C_5\, (x^2)^0 2^5$

$=x^{10} + 20x^8 +160x^6 +640x^4 +1280x^2 +1024$

## General Term in Binomial Expansion

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k$$

The First term would be = ${}^nC_0\, x^n a^0$

The Second term would be = ${}^nC_1\, x^{n-1} a^1$

The Third term would be = ${}^nC_2\, x^{n-2} a^2$

The Fourth term would be = ${}^nC_3\, x^{n-3} a^3$

Like wise (k+1) term would be

$T_{k+1} = {}^nC_k\, x^{n-k} a^k$

This is called the general term also as every term can be find using this term

$T_1 = T_{0+1} = {}^nC_0\, x^n a^0$

$T_2 = T_{1+1} = {}^nC_1\, x^{n-1} a^1$

Similarly for

$$(x-a)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n-k} a^k$$

$T_{k+1} = {}^nC_k\, (-1)^k\, x^{n-k} a^k$

Again similarly for

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$T_{k+1} = {}^nC_k\, x^k$

Again similarly for

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

$T_{k+1} = {}^nC_k\, (-1)^k\, x^k$

**To summarize it**

| Binomial term | (k+1) term |
|---|---|
| $(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k$ | $T_{k+1} = {}^nC_k\, x^{n-k} a^k$ |
| $(x-a)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{n-k} a^k$ | $T_{k+1} = {}^nC_k\, (-1)^k\, x^{n-k} a^k$ |
| $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ | $T_{k+1} = {}^nC_k\, x^k$ |
| $(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$ | $T_{k+1} = {}^nC_k\, (-1)^k\, x^k$ |

## Middle Term in Binomial Expansion

A binomial expansion contains (n+1) terms

If n is even then the middle term would [(n/2)+1 ]th term

If n is odd,then $(n+1)/2$ and $(n+3)/3$ are the middle term

**Examples:**

If the coefficient of $(2k + 4)$ and $(k - 2)$ terms in the expansion of $(1+x)^{24}$ are equal then find the value of k

**Solution:**

The general term of $(1 + x)^n$ is

$T_{k+1} = {}^nC_k\, x^k$

Hence coefficient of $(2k + 4)^{th}$ term will be

$T_{2k+4} = T_{2k+3+1} = {}^{24}C_{2k+3}$

and coefficient or $(k - 2)^{th}$ term will be

$T_{k-2} = T_{k-3+1} = {}^{24}C_{k-3}$

As per question both the terms are equal

${}^{24}C_{2k+3} = {}^{24}C_{k-3}$.

$\Rightarrow (2k + 3) + (k-3) = 24$

$\therefore \quad r = 8$

SUMMARY

# CHAPTER 8

## BINOMIAL THEOREM

Binomial theorem for any positive integer n

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \ldots\ldots + {}^nC_nb^n$$

Recall

1) ${}^nC_r = \dfrac{n!}{(n-r)!\ r!}$
2) ${}^nC_r = {}^nC_{n-r}$

   ${}^7C_4 = {}^7C_3 = \dfrac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$

   ${}^8C_6 = {}^8C_2 = \dfrac{8 \times 7}{1 \times 2} = 28$
3) ${}^nC_n = {}^nC_0 = 1$
4) ${}^nC_1 = n$

**OBSERVATIONS/ FORMULAS**

1) The coefficients ${}^nC_r$ occurring in the binomial theorem are known as binomial coefficients.
2) There are (n+1) terms in the expansion of $(a+b)^n$ , ie one more than the index.
3) The coefficient of the terms equidistant from the beginning and end are equal.
4) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots\ldots + {}^nC_nx^n$. (By putting a = 1 and b = x in the expansion of $(a + b)^n$).
5) $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + \ldots\ldots + (-1)^n\ {}^nC_nx^n$ (By putting a = 1 and b = -x in the expansion of $(a + b)^n$).
6) $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n$ (By putting x = 1 in (4))
7) $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots +(-1)^n\ {}^nC_n.$ (By putting x = 1 in (5))

8**) $(r + 1)^{th}$ term in the binomial expansion for $(a+b)^n$ is called the general term which is given by

$$\mathbf{T_{r+1} = {}^nC_r\ a^{n-r}\ b^r.}$$

i.e to find $4^{th}$ term = $T_4$, substitute r = 3.

9*) **Middle term** in the expansion of $(a+b)^n$

i) If **n is even**, middle term = $\left[\frac{n}{2}+1\right]^{th}$ term.

) If **n is odd**, then 2 middle terms are, $\left[\frac{n+1}{2}\right]^{th}$ term and $\left[\frac{n+1}{2}+1\right]^{th}$ term.

10*) To find the **term independent of x or the constant term,** find the coefficient of $x^0$.(ie put power of x = 0 and find r)

**Problems**

eg 4** (4 marks)

**Ex 8.1**

Q 2,4,7,9 (1 mark)

10*, 11*,12* (4 marks)

13**,14** (4 marks)

13**) Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer

Or

$3^{2n+2} - 8n - 9$ is divisible by 64

Solution: $9^{n+1} - 8n - 9 = (1+8)^{n+1} - 8n - 9$

$= {}^{n+1}C_0 + {}^{n+1}C_1 8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \ldots\ldots + {}^{n+1}C_{n+1} 8^{n+1} - 8n - 9$

$= 1 + 8n + 8 + 8^2 [\, {}^{n+1}C_2 + {}^{n+1}C_3.8 + \ldots\ldots + 8^{n-1}] - 8n - 9$

(since ${}^{n+1}C_0 = {}^{n+1}C_{n+1} = 1$, ${}^{n+1}C_1 = {}^{n+1}$,

$8^{n+1}/8^2 = 8^{n+1-2} = 8^{n-1}$)

$= 8^2 [\, {}^{n+1}C_2 + {}^{n+1}C_3.8 + \ldots\ldots + 8^{n-1}]$ which is divisible by 64

**Problems**

eg 5*,6**,7* (4 marks)

**Ex 8.2**

Q 2,3* (1 mark)

Q 7**,8**,9**,11**,12** (4 marks), 10** (6 marks)

eg 10**,11 (HOT),12 (HOT), 13(HOT), eg 15*,17** (4 marks)

**Misc ex**

Q 1** (6 mark),2,3(HOT), 8* (4 marks)

**Ex 8.2**

Q 10**(6 marks)

The coefficients of the $(r-1)^{th}$ , $r^{th}$ and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio 1: 3 : 5. Find n and r.

Solution

$T_{r+1} = {}^nC_r x^{n-r}$

$T_r = T_{(r-1)+1} = {}^nC_{r-1} x^{n-r+1}$

$T_{r-1} = T_{(r-2)+1} = {}^nC_{r-2}\, x^{n-r+2}$

Given ${}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r :: 1 : 3 : 5$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3}$$

$$\frac{n!}{(n-r+2)!\,(r-2)!} \div \frac{n!}{(n-r+1)!\,(r-1)!} = \frac{1}{3}$$

$$\frac{(n-r+1)!}{(n-r+2)!} \times \frac{(r-1)!}{(r-2)!} = \frac{1}{3}$$

$$\frac{(n-r+1)!}{(n-r+1)!(n-r+2)} \times \frac{(r-2)!(r-1)}{(r-2)!} = \frac{1}{3}$$

$$\frac{r-1}{n-r+2} = \frac{1}{3}$$

$3r-3 = n-r+2$

$n-4r = -5$ _____(1)

$$\frac{{}^nC_{r-1}}{} = \frac{3}{}$$

simplify as above and get the equation $3n - 8r = -3$ _____(2)
solving (1) and (2) we get
$n = 7$ and $r = 3$.

**EXTRA/HOT QUESTIONS**

1) Using Binomial theorem show that $2^{3n} - 7n - 1$ or $8^n - 7n - 1$ is divisible by 49 where n is a natural number. (4 marks**)
2) Find the coefficient of $x^3$ in the equation of $(1+2x)^6 (1-x)^7$ (HOT)
3) Find n if the coefficient of $5^{th}$, $6^{th}$ & $7^{th}$ terms in the expansion of $(1+x)^n$ are in A.P.
4) If the coefficient of $x^{r-1}$, $x^r$, $x^{r+1}$ in the expansion of $(1+x)^n$ are in A.P. prove that $n^2 - (4r+1)n + 4r^2 - 2 = 0$. (HOT)
5) If $6^{th}$,$7^{th}$,$8^{th}$ & $9^{th}$ terms in the expansion of $(x+y)^n$ are respectively a,b,c &d then show that $\frac{b^2 - ac}{c^2 - bd} = \frac{4a}{3c}$ (HOT)
6) Find the term independent of x in the expansion of $\left[3x^2 - \frac{1}{2x^3}\right]^{10}$ (4 marks*)
7) Using Binomial theorem show that $3^{3n} - 26n - 1$ is divisible by 676. (4 marks**)
8) The $3^{rd}$,$4^{th}$ & $5^{th}$ terms in the expansion of $(x+a)^n$ are 84, 280 & 560 respectively. Find the values of x, a and n. (6 marks**)
9) The coefficient of 3 consecutive terms in the expansion of $(1+x)^n$ are in the ratio 3 : 8 :14. Find n. (6 mark**)
10) Find the constant term in the expansion of $(x-1/x)^{14}$
11) Find the middle term(s) in the expansion of

    i) $\left[\frac{x}{a} - \frac{a}{x}\right]^{10}$ ii) $\left[2x - \frac{x^2}{4}\right]^9$

12) If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$
    Prove that $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n.2^{n-1}$

**Answers**

2) -43
3) n = 7 or 14
6) 76545/8
8) x =1, a=2, n = 7
9) 10
10) -3432
11) i) -252
    ii) $\frac{-63}{32} x^{14}$